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Letter to the Editor

Identification of non-linear effects in rotor systems using recursive QR factorization method

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1. Introduction

There are several non-linear and time variant effects in a rotor-bearing dynamic system. The non-linear characteristics of the rolling element bearing, internal damping, unsymmetrical stiffness of the shaft, cracks on the shaft, and the radial clearance are the typical root causes of these effects [1]. As rotating speeds increase and rotor weights decrease, these non-linear and time variant effects could significantly affect the dynamic characteristics of the rotor system [2–8]. If these non-linear and time variant effects are ignored, the vibration response could lead to incorrect interpretation of the rotor system. This will significantly affect the performance of the rotor vibration control schemes, the accuracy of the diagnosis results, and the performance of any schemes where an accurate rotor model is needed. Therefore, the accurate identification of both the model structure and parameters of a rotor-bearing system, including the non-linear effects, is an important engineering problem.

A large body of literature concentrates on this area. Tiwari and Vyas [9] developed a technique for estimating the non-linear stiffness of rolling element bearings. Imam et al. [10] completed an on-line rotor crack detection and monitoring system. Krodkiewski and Ding [11] found a method for on-site estimation of the alignment of multi-bearing rotor systems. Tasker and Chopra [12] used the rotor stability data to estimate the non-linear damping of the system.

Compared to the parameter estimation, the non-linear model structure identification is quite difficult. An optimal search scheme is usually needed to find an adequate model among all the possible ones. Desrochers and Mohseni [13] used a model set that has a multi-layered combinatorial tree structure. They showed that searching the tree for the optimal n-term model could be done in n stages. Kanjilal et al. [14] developed a method for fast selection of significant variables in linear-in-the-parameter models by using modified orthogonal-triangular factorization (also called QR factorization). The distinctive characteristic of their method is that estimations of

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the parameters are not needed. More recently, Gary et al. [15] applied genetic programming optimization technique to find the best non-linear model structures. All of the above methods only work in offline mode. They can be successfully used to determine the non-linear model structures by off-line computation. However, they are not suitable for on-line applications, such as the real-time supervisory adaptive control schemes, in which the model parameters and/or the model structures are determined during operation.

Niu and Fisher [16] presented a method to detect identifiability problems, such as the non-persistent excitation, over-parameterization, and output feedback within the system, for the on-line identification of AutoRegressive model with eXogeneous inputs (ARX model) using the augmented UD factorization. Their method has the potential to be used for the ARX model structure determination, but it cannot be extended to the identification of a general non-linear (but linear-in-the-parameter) system. The order downdating algorithm in recursive least squares identification developed by Apley and Shi [17] opens the possibility of determining both the structure of a linear-in-the-parameter system and the parameters simultaneously in real time. This method consists of a time updating portion and an order downdating portion. The parameters and residual error energies for an entire set of models can be obtained efficiently.

In this paper, an on-line estimation method that can simultaneously estimate the parameters and determine the significance of the non-linear and time variant effects in the rotor-bearing dynamic system is presented. This method is based on an order downdating algorithm, which can deal with all linear-in-the-parameter non-linear and time variant effects, such as the unsymmetrical shaft, the structural internal damping, and the non-linear elastic restoring force of the bearing.

The equation of motion of a four-degree-of-freedom (d.o.f) rotor is given in Section 2. This dynamic model includes one time variant effect (unsymmetrical shaft) and two non-linear effects (structural internal damping and the non-linear spring restoring force). In Section 3, this dynamic equation is formulated in a form that is suitable for recursive least squares estimation in both timely and orderly sense. A numerical study is given in Section 4. The paper is concluded in Section 5.

2. Modeling of non-linear and time variant effects of rotor system

2.1. Geometric set-up of the rotor

In this research, a four d.o.f rotor as shown in Fig. 1 is considered.

The rotor bearing system is modelled as a rigid disk located at the middle of a massless elastic shaft, which is supported by two rigid bearings. To describe the motion of the rigid disk, three co-ordinate systems are needed. $OXYZ$ frame is the inertial frame. The Z-axis coincides with the rotation axis of the rotor. O is the geometric center of the rotor when it is at rest. $Cuvw$ frame is the body-fixed frame that has the same motion as the disk. $Cxyz$ frame is an intermediate frame between $OXYZ$ frame and $Cuvw$ frame. The $Cxyz$ frame is a non-spinning frame. The orientation of $Cxyz$ is obtained by first rotating $OXYZ$ in the OX direction by θ_y and then rotating in the new OY direction by θ_x. Since θ_y and θ_x are small, the sequence of these two rotations is not important. $Cuvw$ is obtained from $Cxyz$ by rotating in the Cz (also Cw) direction with the spinning

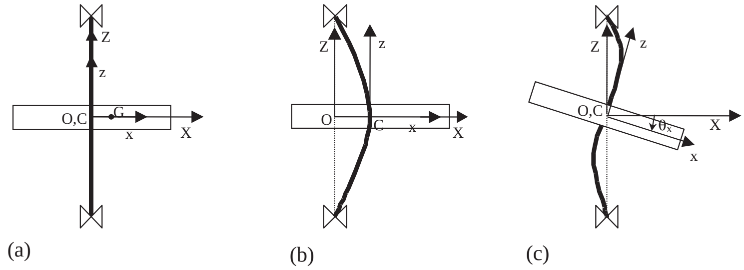

Fig. 1. The geometric setup and two vibration modes of the rotor system: (a) initial condition, (b) deflection motion, (c) inclination motion.

angle ϕ. The rotating motion of the rigid disk about the shaft is predefined. The translational motion of the disk along the OZ direction is neglected. Therefore, the system has four d.o.f.

2.2. Equation of motion

There are two vibration modes in the rotor system: the deflection motion as shown in Fig. 1(b) and the inclination motion as shown in Fig. 1(c). To further simplify the motion, we assume that the disk is at the center of the shaft. Therefore, these two vibration motions are decoupled. In this paper, only the inclination motion will be considered because the gyroscopic effect can be included in the inclination motion. Following the derivation of Genta [18], we can get the rotational kinetic energy of the rotor,

$$\begin{aligned} T_{rot} = \tfrac{1}{2}\,\{&I_t(\dot{\theta}_x^2 + \dot{\theta}_y^2) + I_p(\dot{\phi}^2 + 2\dot{\phi}\theta_x\dot{\theta}_y) \\ &+\ 2\dot{\phi}\tau(I_p - I_t)[\dot{\theta}_y \cos\phi + \dot{\theta}_x \sin\phi]\}. \end{aligned} \tag{1}$$

Taking the θ_x and θ_y as the generalized co-ordinates, the governing equation for the inclination motion is

$$\begin{aligned} I_t\ddot{\theta}_y + I_p\dot{\phi}\dot{\theta}_x &= (I_t - I_p)\tau(\ddot{\phi}\cos\phi - \dot{\phi}^2 \sin\phi) + M_y, \\ I_t\ddot{\theta}_x - I_p\dot{\phi}\dot{\theta}_y &= (I_t - I_p)\tau(\ddot{\phi}\sin\phi + \dot{\phi}^2 \cos\phi) + M_x. \end{aligned} \tag{2}$$

M_y and M_x are the applied external moment along Cx and Cy directions. I_t and I_p are the diametrical and polar moment inertia of the disk. ϕ, $\dot{\phi}$ and $\ddot{\phi}$ are the rotating angle, the angular velocity, and the angular acceleration of the rotor, respectively. τ is the dynamic unbalance of the disk. Based on this equation, we can add several linear and non-linear terms as follows:

- The linear damping terms $-c\dot{\theta}_y$ and $-c\dot{\theta}_x$. These effects are caused by the aerodynamic and other viscous effects.
- The linear spring restoring force of an unsymmetrical stiffness shaft. The unsymmetrical stiffness of the shaft could be due to the crack on the shaft and the unsymmetrical geometry of the shaft. If we assume that the stiffness of the shaft in Cu direction is k_u and in Cv direction is k_v, the elastic restoring force in the rotating frame is just $[-k_u\theta_u \quad -k_v\theta_v]^{\mathrm{T}}$. By multiplying the

co-ordinate transformation matrix

$$\begin{bmatrix} \cos\phi & \sin\phi \\ -\sin\phi & \cos\phi \end{bmatrix}^{\mathrm{T}}$$

between *Cxyz* and *Cuvw*, we can get the restoring force expressed in the non-spinning *Cxyz* co-ordinate. Defining

$$k = \frac{k_u + k_v}{2}, \quad \Delta = \frac{k_u - k_v}{2}$$

and noting

$$\begin{bmatrix} \theta_u \\ \theta_v \end{bmatrix} = \begin{bmatrix} \cos\phi & \sin\phi \\ -\sin\phi & \cos\phi \end{bmatrix} \begin{bmatrix} \theta_y \\ \theta_x \end{bmatrix},$$

we can get

$$\begin{aligned} M_{ky} &= -k\theta_y - \Delta(\theta_y \cos 2\phi + \theta_x \sin 2\phi), \\ M_{kx} &= -k\theta_x - \Delta(\theta_y \sin 2\phi - \theta_x \cos 2\phi), \end{aligned} \tag{3}$$

where M_{ky} and M_{kx} are the linear restoring moments along *Cx* and *Cy* directions.

- The non-linear elastic restoring moment, represented as a second order polynomial [3],

$$\begin{aligned} N_y &= -(k_{03}\theta_y^2 + 2k_{12}\theta_x\theta_y + k_{21}\theta_x^2), \\ N_x &= -(k_{30}\theta_x^2 + 2k_{21}\theta_x\theta_y + k_{12}\theta_y^2), \end{aligned} \tag{4}$$

where N_y and N_x are the non-linear restoring moments along *Cx* and *Cy* directions.

- Internal damping. There are two kinds of internal damping: material damping and structural damping. For a steel shaft, the structural damping is important. It is caused by dry friction between the elastic shaft and the inner ring of the rigid disk. It can be simplified as a constant magnitude moment in the opposite direction of the relative angular velocity. Since this friction is between the rotating parts, this force can be easily expressed in the rotating frame as $\mathbf{f} = -h\dot{\mathbf{v}}/|\dot{\mathbf{v}}|$. $\mathbf{v}$ is the relative angular motion between the rotating shaft and rigid disk. h is a constant coefficient. To put this term in Eq. (2), $\mathbf{f}$ must be expressed in the non-spinning frame *Cxyz*. If viewing a vector $\mathbf{v}$ as a complex number, then the transformation from $\mathbf{v}$ expressed in the body-fixed frame *Cuvw* to $\mathbf{v}_{ns}$ expressed in the non-spinning frame *Cxyz* is $\mathbf{v}_{ns} = \mathbf{v}\mathrm{e}^{\mathrm{i}\phi(t)}$. Therefore, $\mathbf{v} = \mathbf{v}_{ns}\mathrm{e}^{-\mathrm{i}\phi(t)}$ and $\dot{\mathbf{v}} = \dot{\mathbf{v}}_{ns}\mathrm{e}^{-\mathrm{i}\phi(t)} - \mathrm{i}\mathbf{v}_{ns}\dot{\phi}(t)\mathrm{e}^{-\mathrm{i}\phi(t)}$ We can get

$$\mathbf{f}_{ns} = -h\,\frac{\dot{\mathbf{v}}_{ns}\mathrm{e}^{-\mathrm{i}\phi(t)} - \mathrm{i}\mathbf{v}_{ns}\dot{\phi}(t)\mathrm{e}^{-\mathrm{i}\phi(t)}}{|\dot{\mathbf{v}}_{ns}\mathrm{e}^{-\mathrm{i}\phi(t)} - \mathrm{i}\mathbf{v}_{ns}\dot{\phi}(t)\mathrm{e}^{-\mathrm{i}\phi(t)}|}\,\mathrm{e}^{\mathrm{i}\phi(t)} = -h\,\frac{\dot{\mathbf{v}}_{ns} - \mathrm{i}\mathbf{v}_{ns}\dot{\phi}(t)}{|\dot{\mathbf{v}}_{ns} - \mathrm{i}\mathbf{v}_{ns}\dot{\phi}(t)|}. \tag{5}$$

Noting $\mathbf{v}_{ns} = \theta_y + \mathrm{i}\theta_x$ and $\dot{\mathbf{v}}_{ns} = \dot{\theta}_y + \mathrm{i}\dot{\theta}_x$, the structural internal damping terms are

$$\begin{aligned} M_{dy} &= -h\,\frac{(\dot{\theta}_y + \dot{\phi}\theta_x)}{\sqrt{\dot{\theta}_x^2 + \dot{\theta}_y^2 + 2\dot{\phi}(\dot{\theta}_y\theta_x - \dot{\theta}_x\theta_y) + \dot{\phi}^2(\theta_x^2 + \theta_y^2)}}, \\ M_{dx} &= -h\,\frac{(\dot{\theta}_x - \dot{\phi}\theta_y)}{\sqrt{\dot{\theta}_x^2 + \dot{\theta}_y^2 + 2\dot{\phi}(\dot{\theta}_y\theta_x - \dot{\theta}_x\theta_y) + \dot{\phi}^2(\theta_x^2 + \theta_y^2)}}, \end{aligned} \tag{6}$$

where M_{dy} and M_{dx} are the non-linear internal damping moment along the Cx and Cy directions.

Substituting all these terms into Eq. (2), the overall governing equation can be obtained as

$$I_t\ddot{\theta}_y + I_p\dot{\phi}\dot{\theta}_x + c\dot{\theta}_y - M_{dy} - N_y - M_{ky} = (I_t - I_p)\tau(\ddot{\phi}\cos\phi - \dot{\phi}^2\sin\phi),$$
$$I_t\ddot{\theta}_x - I_p\dot{\phi}\dot{\theta}_y + c\dot{\theta}_x - M_{dx} - N_x - M_{kx} = (I_t - I_p)\tau(\ddot{\phi}\sin\phi + \dot{\phi}^2\cos\phi). \quad (7)$$

The above equation can be changed to a dimensionless form, by defining the dimensionless variables be $t' = t\sqrt{k/I_t}$, $i_p = I_p/I_t$, $c' = c/\sqrt{kI_t}$, $\Delta' = \Delta/k$, $k'_{ij} = k_{ij}/k$, and $h' = h/k$,

$$\ddot{\theta}_y + i_p\dot{\phi}\dot{\theta}_x + c'\dot{\theta}_y - M'_{dy} - N'_y - M'_{ky} = (1 - i_p)\tau(\ddot{\phi}\cos\phi - \dot{\phi}^2\sin\phi),$$
$$\ddot{\theta}_x - i_p\dot{\phi}\dot{\theta}_y + c'\dot{\theta}_x - M'_{dx} - N'_x - M'_{kx} = (1 - i_p)\tau(\ddot{\phi}\sin\phi + \dot{\phi}^2\cos\phi), \quad (8)$$

where M'_{dy}, M'_{dx}, N'_y, N'_x, M'_{ky}, and M'_{kx} are obtained by replacing the dimensional variables with the dimensionless variables in Eqs. (3)–(6).

The model developed in this section includes unsymmetrical stiffness, structural internal damping, non-linear elastic restoring force, and gyroscopic effects. These effects are all linear in the parameters. Therefore, these parameters can be estimated by the QR factorization method. Compared to the ordinary linear regression algorithm, the QR factorization method is more numerically stable. In addition, the QR method can give the estimation of parameters and the corresponding residual energies of a set of models simultaneously. Using this information, we can determine the significance of the non-linear and time variant effects, which are usually not equally significant in a given rotor system.

3. On-line identification of non-linear and time varying effects using QR factorization

3.1. Linear regression formulation of the rotor dynamic model

A linear-in-the-parameter model can be estimated by linear regression. The two equations in Eq. (8) are symmetric. For the determination of non-linear and time varying effects, only one equation is needed. If we gather the known terms $(\ddot{\theta}_y + i_p\dot{\phi}\dot{\theta}_x + \theta_y)$ and the rest of the unknown terms in the first equation of Eq. (8), the dynamic system can be formulated into a linear regression format. Denoting $y = -(\ddot{\theta}_y + i_p\dot{\phi}\dot{\theta}_x + \theta_y)$, the regressor as

$$\mathbf{x} = [x_1, x_2, x_3, x_4, x_5, x_6, x_7]^{\mathrm{T}}, \quad (9)$$

where $x_1 = (i_p - 1)(\ddot{\phi}\sin\phi + \dot{\phi}^2\cos\phi)$, $x_2 = (\theta_y\cos 2\phi + \theta_x\sin 2\phi)$, $x_3 = \dot{\theta}_y$, $x_4 = \theta_x^2$, $x_5 = 2\theta_x\theta_y$, $x_6 = \theta_y^2$,

$$x_7 = \frac{(\dot{\theta}_y + \dot{\phi}\theta_x)}{\sqrt{\dot{\theta}_x^2 + \dot{\theta}_y^2 + 2\dot{\phi}(\dot{\theta}_y\theta_x - \dot{\theta}_x\theta_y) + \dot{\phi}^2(\theta_x^2 + \theta_y^2)}}$$

and the unknown parameters as

$$\varphi = [\tau, \Delta', c', k'_{21}, k'_{12}, k'_{03}, h']^{\mathrm{T}}, \quad (10)$$

we can obtain

$$y = \mathbf{x}^{\mathrm{T}}\varphi + \varepsilon. \tag{11}$$

There are several assumptions in the above linear regression formulation.

1. The rotating angle, speed, and the acceleration of the rotor, i.e., ϕ, $\dot{\phi}$, and $\ddot{\phi}$ are known. The rotating speed can be easily obtained from the tachometer. The rotating angle and the rotating acceleration can be obtained by numerical integration and differential method.
2. The diametrical and the polar moment of inertia are known. These variables can be calculated from the geometric configuration of the rotor.
3. The full states of the rotor system are measurable. That means that θx, θy, and corresponding angular velocities and accelerations are known. The inclination angles and angular velocities can be obtained by non-contact displacement and velocity sensors. The angular acceleration can be obtained by a numerical differential method.
4. The symmetrical stiffness k is known. If it is unknown, we cannot use the dimensionless dynamic equations (8). Instead, we have to use Eq. (7) to formulize the linear regression equation and take k as an unknown parameter. The following method can still be used.
5. The term ε is a normally distributed white noise. This noise has several sources, such as modelling error, measurement error, and numerical round error. Since this noise consists of several noise sources, we assume that it has normal distribution in this paper.

Many methods can be used to solve this linear regression problem [19]. Among these methods, the QR factorization method enjoys numerical stability and inherited orderly recursive ability.

3.2. QR factorization method

One form of QR factorization of an m-by-n matrix is given by $\mathbf{A} = \mathbf{QR}$, where $\mathbf{Q}$ is an m-by-n orthogonal matrix and $\mathbf{R}$ is an n-by-n upper triangular matrix with ones on the diagonal. The QR factorization is closely related with the least squares estimation. Define J_i as the ith diagonal element of the diagonal matrix $\mathbf{Q}^{\mathrm{T}}\mathbf{Q}$ and

$$\mathbf{R}^{-1} = \begin{bmatrix} 1 & \boldsymbol{\varphi}_2 & \boldsymbol{\varphi}_3 & \cdots & \boldsymbol{\varphi}_n \\ & 1 & & & \\ & & 1 & & \\ & 0 & & \ddots & \\ & & & & 1 \end{bmatrix},$$

where $\boldsymbol{\varphi}_i$ is a column vector of length $i-1$. It is well known that $\boldsymbol{\varphi}_i$ is the negative of the least squares coefficient vector in projecting the ith column of matrix $\mathbf{A}$ onto the space spanned by the previous $i-1$ columns of $\mathbf{A}$. The corresponding sum of squares of residue error (residue energy) is given by J_i [20]. Mathematically,

$$\boldsymbol{\varphi}_i = -\arg\min_{\boldsymbol{\varphi}} \|\mathbf{a}_i - \mathbf{A}^{i-1}\boldsymbol{\varphi}\|_2, \quad \mathbf{q}_i = \mathbf{a}_i + \mathbf{A}^{i-1}\boldsymbol{\varphi}_i, \quad \text{and} \quad J_i = \mathbf{q}_i^{\mathrm{T}}\mathbf{q}_i, \tag{12}$$

where $\mathbf{a}_i$ is the ith column of matrix $\mathbf{A}$ and $\mathbf{A}^{i-1}$ is a sub-matrix consisting of first $i-1$ columns of $\mathbf{A}$, $\mathbf{q}_i$ is the ith column of $\mathbf{Q}$. Using this property, the linear regression problem can be solved by QR factorization if we let $\mathbf{A} = [\mathbf{x}^{\mathrm{T}}\ y]$.

The QR factorization can be obtained by using the modified Gram–Schmidt orthogonalization [21], which is more numerically stable than the classical Gram–Schmidt orthogonalization procedure. Ling et al. [22] found a timely recursive scheme to conduct the modified Gram–Schmidt procedure. For an m-by-n matrix $\mathbf{A}_t$ and an $(m+1)$-by-n matrix $\mathbf{A}_{t+1} = [\mathbf{A}_t^{\mathrm{T}}\ \ \mathbf{b}^{\mathrm{T}}]^{\mathrm{T}}$ where $\mathbf{b}$ is a new 1-by-n row vector (a new observation in Eq. (11)), the QR factorization of matrix $\mathbf{A}_{t+1}$ can be recursively obtained from the QR factorization of matrix $\mathbf{A}_t$. Sakai [23] extended Ling's algorithm to the inverse QR factorization method in which $\mathbf{R}^{-1}$ is updated directly. The parameters of least squares estimation $\boldsymbol{\varphi}_i$ can be updated directly. This method can be viewed as a complete recursive least squares estimation method with orderly recursive capability because we can also obtain the projection of the intermediate column of $\mathbf{A}$ onto the space spanned by the previous columns.

3.3. Identification of non-linear and time-varying effects in rotor system

To determine the significance of a non-linear or time variant effect, we need to fit a model with and without that certain effect. For example, to determine the effect of non-linear structural damping, we need to map the response y onto the space spanned by $\mathbf{x}$ (i.e., finding least squares solution of Eq. (11)) and map the response y onto the space spanned by $\mathbf{x}' = [x_1, x_2, x_3, x_4, x_5, x_6]^{\mathrm{T}}$, which represents the system without the non-linear strcutural damping force. Then, these two models are compared in a statistical sense to determine if this non-linear effect is significant.

Apley and Shi [17] extended the recursive inverse QR factorization algorithm by adding a recursive orderly updating part in the original timely updating scheme. Their algorithm can be summarized as follows. Given the inverse QR factorization of m-by-$(n+1)$ matrix $[\mathbf{x}_1\mathbf{x}_2\ldots\mathbf{x}_i\ \mathbf{y}\ \mathbf{x}_{i+1}\ldots\mathbf{x}_n]$, where $\mathbf{x}_i$ $(i = 1,\ldots,n)$ and $\mathbf{y}$ are m-by-1 column vectors, the inverse QR factorization of matrix $[\mathbf{x}_1\ \mathbf{x}_2\ \ldots\mathbf{x}_{i-1}\ \mathbf{y}\ \mathbf{x}_i\ \mathbf{x}_{i+1}\ldots\mathbf{x}_n]$ can be obtained recursively. Together with the timely updating part, we can get a fully recursive order downdating least squares estimation algorithm. The structure of these n models are shown in Fig. 2. The ith model is of order i. The basic steps of this

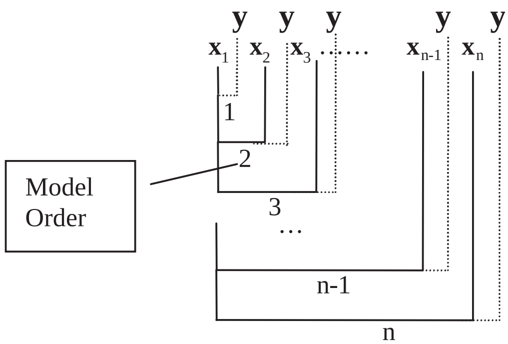

Fig. 2. The structures of n models estimated by the QR method.

algorithm are listed as follows (the details can be found in [17]):

1. Timely update R−1(t), J(t), and the last row of Q(t). The estimated least squares parameters are in R−1(t). The matrix J(t) is the diagnoal matrix QT(t)Q(t). The elements of J(t) are the residual error energies of the corresponding model.
2. Perform order downdating. Based on the results of step 1, we can obtain the parameters and residual error energies of model n to model 1 as shown in Fig. 2.
3. Output the interested parameters and residual error energies.

This algorithm can efficiently calculate the least squares parameters and the corresponding residual error energies of n models.

In the rotor senerio, the statistical significance of the non-linear and time variant effects can be determined by statistical model selection rules, such as the partial F-test [19], the AIC criterion, or FPE test [24] on the residual error energies. These statistic criteria have similar performance. In this paper, the partial F-test method is adopted.

Suppose that there are two models, Model I and Model II. Model I is a special case of Model II, i.e.,

$$\text{Model I:}\quad \mathbf{y} = \beta_1\mathbf{x}_1 + \cdots + \beta_q\mathbf{x}_q + \varepsilon;$$

$$\text{Model II:}\quad \mathbf{y} = \beta_1\mathbf{x}_1 + \cdots + \beta_q\mathbf{x}_q + \beta_{q+1}\mathbf{x}_{q+1} + \cdots + \beta_p\mathbf{x}_p + \varepsilon'.$$

For testing the significance of the contributions of the factors $\mathbf{x}_{q+1}, \ldots, \mathbf{x}_p$, or equivalently, for testing the adequacy of the Model I, the null hypothesis $H_0 : \beta_{q+1} = \cdots = \beta_p = 0$ should be tested, which can be done by checking the F-statistic,

$$\frac{(Residual\ \ Error\ \ Energy\ \ of\ \ Model\ \ I - Residual\ \ Error\ \ Energy\ \ of\ \ Model\ \ II)/(p-q)}{Residual\ \ Error\ \ Energy\ \ of\ \ Model\ \ II/(m-p)}, \tag{13}$$

where m is the sampling size. By calculating the F-statistic, and then computing its p-value, it can be determined whether the null hypothesis should be rejected. If so, that means Model I is not adequate, or, in other word, the factors $\mathbf{x}_{q+1}, \ldots, \mathbf{x}_p$ are significant. On the other hand, if we cannot reject the null hypothesis, that means the factors $\mathbf{x}_{q+1}, \ldots, \mathbf{x}_p$ are not significant.

Confined by the on-line real-time calculation, only n successive order increasing models are available by QR estimation as shown in Fig. 2. Hence, the heuristic searching method, such as the forward successive model selection or backward successive model selection method used by Draper and Smith [19], cannot be used with the QR method. To use the F-test based on these n models, the sequence of factors, $x_1 \sim x_n$, in QR estimation should be carefully selected. This sequence determines that the F-test is performed on which n models among all $2^n - 1$ possible models. The selection of the sequence (hence, the n models) depends on the characteristic of the physical system and is determined by our interests. In most cases, it is very difficult, if not impossible, to select these n models purely based on theoretical analysis. Empirical studies are often needed to determine the sequence of these factors. For example, if we want to use this method for process monitoring and fault detection, then roughly speaking, the significant factors under normal working condition of the rotor system should be put in the front part of the sequence as the core factors. When a fault happens in the system, certain factors may become significant (e.g., a crack could suddenly happen in the shaft. In this case, the un-symmetric of shaft will become significant). These factors, based on the probability of happening, should be put after the core factors. The factors of the high probability faults should be put in front of the low probability factors. In this way, the non-linear effect identification actually can be used as a system fault detection system. The focus of this paper is to show the capability of recursive QR factorization estimation and the F-test method. Therefore, in this context, we will simply select the sequence of the factors as shown in Eq. (10).

The procedure for the determination of the significance of the non-linear and time variant effects and the corresponding parameters is summarized as follows:

- Selecting a sequence of the factors based on the physical characteristics of the system and the interests.
- Initializing the orderly recursive estimation algorithm. The initial values of the parameters ($\boldsymbol{\varphi}$ in Eq. (11)) can all be zeros. To avoid being divided by zero, the initial values of the error energies should be small positive numbers.
- Measuring and calculating y and all the elements of $\mathbf{x}$ in Eq. (11).
- Using the order downdating recursive least squares estimation algorithm [17] to calculate the estimated parameters of a series of models and the corresponding residual error energies.
- Invoking statistical model selection criteria to determine the significance of each non-linear and time variant effect.

In next section, a numerical study is conducted to show the effectiveness of this method.

4. Numerical study

The objective of this simulation study is to illustrate the procedure developed in Section 3 for the parameter estimation and the model structure determination of the rotor dynamic system.

The dynamic system Eq. (8) are solved by the Runge–Kutta method to obtain the motion of the system. A normal distributed noise with variance 10^{-4} is added to the system output y. The rotor is at rest at time 0 and accelerates at a constant angular rate of 0.01. The rotating speed of the rotor is the same as the natural frequency of the rotor system at dimensionless time 100. Other parameters used in the simulation are listed in Table 1. Since a dimensionless model is adopted to capture the essential dynamic characteristics of the system, these numbers are dimensionless. One dimensionless model corresponds to multiple physical models. For example, the dimensionless parameters used in this numerical study can correspond to a physical system: $k = 3.75 \times 10^6\,\mathrm{kg\,m^2/s^2}$, $i_t = 1500\,\mathrm{kg\,m^2}$, $i_p = 450\,\mathrm{kg\,m^2}$, $c = 2250\,\mathrm{kg\,m^2/s}$, $\tau = 0.02$, $h = 3.75 \times 10^4\,\mathrm{kg\,m^2/s^2}$, $\Delta = 112\,500\,\mathrm{kg\,m^2/s^2}$, k_{03}, k_{12}, k_{30}, and k_{21} are 7.5×10^4, 1.125×10^5, 1.875×10^5, and $1.5 \times 10^5\,\mathrm{kg\,m^2/s^2}$, respectively. The acceleration of the rotating motion is 1500 r.p.m./s. The critical speed of the system is 3000 r.p.m.

Table 1
Parameters used in the numerical study

i'_p	c'	τ	h'	Δ'	k'_{03}	k'_{12}	k'_{30}	k'_{21}
0.3	0.03	0.02	0.01	0.03	0.02	0.03	0.05	0.04

The rationale of considering the acceleration case rather than the constant rotating speed case lays in the fact that the maximum vibration magnitude usually occurs at the time when the rotating speed hits the critical speed of the rotor. Therefore, if we try to minimize the maximum vibration of the rotor by passive or active control scheme, we have to consider the non-stationary vibration period when the rotor passes its critical speed. An accurate model is important for the design of an efficient passive or active control scheme. Therefore, in this study, we only consider the acceleration period. Based on these parameters and initial conditions, the response of rotor is shown in Fig. 3.

The simulation shows that the vibration magnitude is very small before time step 100 (at time 100, the rotor spins at its natural frequency). The vibration magnitude increases dramatically between time 100 and 150. This simulation coincides with the phenomena observed in the experiment that the resonant peaks usually arrive when the rotating speed of the rotor is higher than the natural frequency during acceleration [25]. After the resonant peak at about 150, the magnitude of the vibration slowly diminishes. The whole response is very similar to the response of a mass–spring linear system. This is true because the non-linear and damping terms are very small compared with the symmetrical linear stiffness (1 in the dimensionless equation) in this study.

Formulating the linear regression problem and using the order downdating QR estimation algorithm, we can get the parameter estimation and the corresponding residual error energies. The initial guess of the parameters are all zeros and the initial J_i's estimation algorithm are all selected as 0. 01. The estimation results for the full model that includes all the effects in Eq. (11) are shown in Fig. 4.

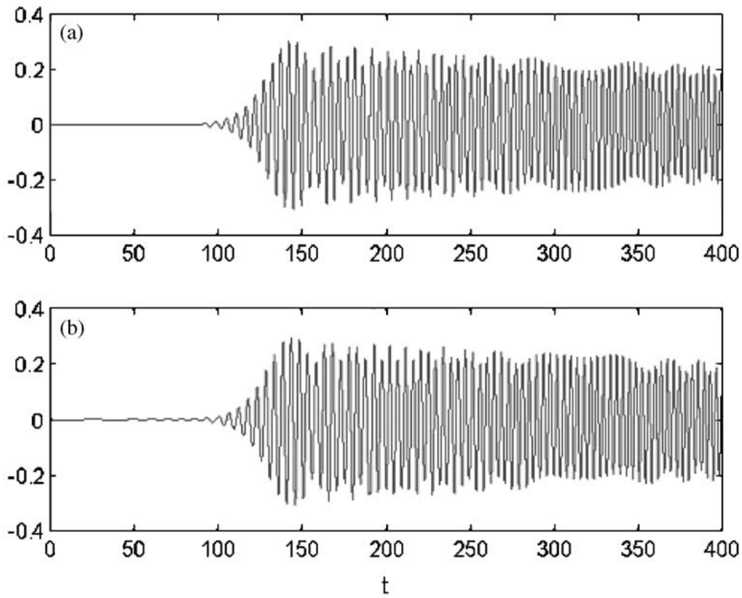

Fig. 3. The response of rotor during acceleration: (a) θ_x and (b) θ_y.

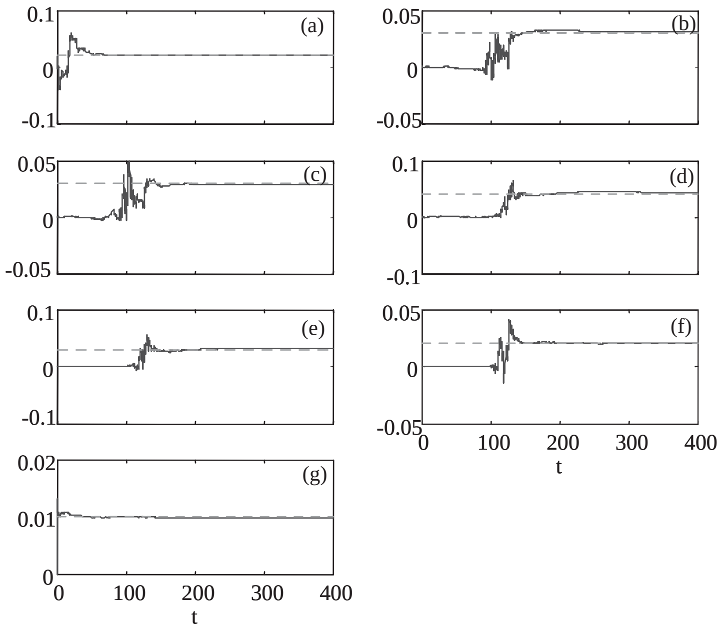

Fig. 4. The least squares parameters estimation: (a) τ, (b) Δ', (c) c', (d) k'_{21}, (e) k'_{12}, (f) k'_{03}, and (g) h'. The dashed lines are the true values.''

One significant characteristic of these results is that the parameter estimations do not converge to the true value before the rotating speed of the rotor hits the natural frequency. There are two reasons: First, when the rotating speed is below the resonant peak the vibration is so small that the output signal is dominated by noise. The second fact is that the natural vibration mode is not excited before the resonant peak occurs. That means this system cannot be distinguished from a simple first order system in terms of the input and output of the system. Therefore, the estimated parameter is different from the true parameters before the dimensionless time 150. (In the simulation, the rotating speed hits the natural freqency at 100. However, the fast accleration pushes the resonant peak to happen at a higher speed.)

The partial F-test is done based on this recursive orderly downdating QR algorithm. The results are shown in Fig. 5. The horizontal axis represents dimensionless time. The vertical axis represents the results of the F-test. The results are given as a binary value: one means the corresponding factor is significant and zero means the corresponding factor is not significant. The α error is selected as 0.05 in the F-test. Each figure in Fig. 5 shows the result of partial F-statistics between two models. For example, the first figure in Fig. 5 shows the results of the partial F-statistics between model 1 and model 2. The structure of models 1 and 2 are shown in Fig. 2. In more detail, model 1 is $\mathbf{y} = \mathbf{x}_1^{\mathrm{T}}\varphi_{11} + \varepsilon_1$ and model 2 is $\mathbf{y} = \mathbf{x}_1^{\mathrm{T}}\varphi_{21} + \mathbf{x}_2^{\mathrm{T}}\varphi_{22} + \varepsilon_2$. This test can show the statistical significance of the effect of asymmetric stiffness of the rotor.

Since the estimation is done recursively at each time step, the F-statistic is calculated at each time step using Eq. (13). The simulation results have several notable characteristics. First, almost no dynamic effects are significant before the resonant peak. Clearly, the reason is that the vibration is so small before the resonant peak that the noise is dominant in the output y. This is

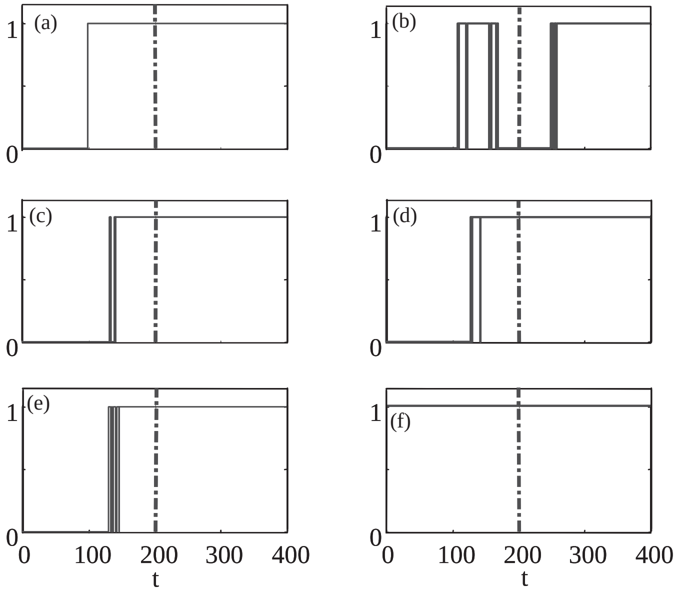

Fig. 5. The partial F-statistic tests for the model including all the effects. Zero means insignificant and one means significant. The dashed line is at $t = 200$. Each subfigure shows the test result for a specific factor: (a) Δ', (b) c', (c) k'_{21}, (d) k'_{12}, (e) k'_{03}, (f) h'.

consistent with the parameter estimation result. Both the parameter estimation and the significance test are not accurate before the resonant peak happens. Second, there are some oscillations in the results between 100 and 200 s. These oscillations are due to the transient response of the estimator. The result after $t = 200$ should be used in this case. Third, the significance of linear damping effect c' cannot be identified correctly even after $t = 200$. This is caused by the negative correlation between the linear damping and the non-linear structural damping. The partial F-test between Models 2 and 3, which only includes the linear damping, tends to underestimate the significance of the linear damping. However, since our interests are on the non-linear effects, this underestimation will not affect the application of this method.

A numerical study without the structural damping (letting $h' = 0$ in the simulation) has also been done. Because of the lack of structural damping, the significance of linear damping can be identified. Similar numerical studies have been done for the significance testing of the model without asymmetric stiffness (letting $\Delta' = 0$) and the model without one of the non-linear spring restoring forces (letting k'_{12} equals 0). These numerical results are shown in Table 2.

The second column of this table shows the true parameters. Since QR method is an on-line recursive method, we can obtain the parameter estimation, F-statistics for each parameter, and the parameter significance at each time step. The values in the third to fifth column are just typical values of these after $t = 200$. The symbol "—" in the significance column means that the significance cannot be decided. From these numerical studies, we can see that this method can efficiently determine the significance of the non-linear and time varying factors in rotor system after the resonant peak occurs. To use this method, we need to know when the resonant peaks

Table 2
Results of the numerical study

Case no.	True parameters	Estimated parameters		F-statistics		Significance	
1	As shown in Table 1	τ	0.0201				
		Δ'	0.0321	Δ'	809.73	Δ'	Yes
		c'	0.0292	c'	0.52079	c'	—
		k'_{21}	0.0426	k'_{21}	129.67	k'_{21}	Yes
		k'_{12}	0.0297	k'_{12}	67.693	k'_{12}	Yes
		k'_{03}	0.0205	k'_{03}	67.733	k'_{03}	Yes
		h'	0.0098	h'	4709.8	h'	Yes
2	As shown in Table 1 except $h' = 0$	τ	0.0201				
		Δ'	0.0320	Δ'	723.98	Δ'	Yes
		c'	0.0285	c'	1292.2	c'	Yes
		k'_{21}	0.0421	k'_{21}	99.92	k'_{21}	Yes
		k'_{12}	0.0273	k'_{12}	24.502	k'_{12}	Yes
		k'_{03}	0.0214	k'_{03}	93.005	k'_{03}	Yes
		h'	−0.0002	h'	1.7157	h'	No
3	As shown in Table 1 except $\Delta' = 0$	τ	0.0201				
		Δ'	0.0022	Δ'	1.2566	Δ'	No
		c'	0.0294	c'	2.4175	c'	—
		k'_{21}	0.0430	k'_{21}	315.6	k'_{21}	Yes
		k'_{12}	0.0296	k'_{12}	134.83	k'_{12}	Yes
		k'_{03}	0.0206	k'_{03}	182.84	k'_{03}	Yes
		h'	0.0099	h'	5242.6	h'	Yes
4	As shown in Table 1 except $k'_{12} = 0$	τ	0.0199				
		Δ'	0.0292	Δ'	1837.5	Δ'	Yes
		c'	0.0286	c'	10.473	c'	—
		k'_{21}	0.0392	k'_{21}	301.07	k'_{21}	Yes
		k'_{12}	−0.0020	k'_{12}	0.023104	k'_{12}	No
		k'_{03}	0.0195	k'_{03}	195.92	k'_{03}	Yes
		h'	0.0098	h'	5319.1	h'	Yes

occur to avoid unreliable results. This is not difficult because large non-synchronous vibration will happen at resonant peaks.

Another point that needs to be emphasized is that the significance of the factors is in the statistical sense. They can be viewed as the energy of the portion in the output that is contributed by certain factors. This significance does not consider specific application. A statistically significant factor could be insignificant with respect to a certain application. For example, factor A could contribute a large portion of the output. Therefore, it is statistically significant. However, if there is a control scheme that only suppresses the output within a certain frequency range and the range does not include the output caused by factor A, this factor will be insignificant for this control system. A practical example is the active balancing system that only controls the synchronous vibration of the rotor. The whole vibration signal will be filtered by a band-pass filter before it is fed to the control algorithm. Therefore, all factors that contribute non-synchronous vibration, such as the non-linear spring restoring forces, will be insignificant. How to determine the significance of the factors with respect to specific application will be studied in the future.

5. Conclusion

A rotor is a primary part of many mechanical systems, making rotor vibration control an important engineering problem. In general, non-linear and time varying effects can affect the dynamics of the rotor significantly in certain rotor systems. This will cause more difficulties in the active vibration control for the rotor system because the active vibration scheme usually assumes that the underlay plant is linear [26,27]. In this paper, the on-line estimation of parameters and the determination of the significance of several non-linear and time varying linear effects in rotor dynamic systems are investigated.

After building a dynamic model of the rotor that includes both linear and non-linear time varying effects, QR factorization method is used to estimate the parameters of these effects. The study shows that the QR factorization method for the least squares estimation is numerically stable. This can be shown by the simulation before the resonant peak. At that time, the signal is so small that the noise dominates the whole output. However, the oscillation of the QR estimation is small and the estimation converges to the true value right after the resonant peak. By using the orderly recursive ability of the QR factorization method, n selected models among all possible $2^n - 1$ models (if there are n factors and one output) can be obtained at each time step. The statistical significance of various non-linear or time varying factors can be obtained by partial F-statistic test if these n models are carefully selected using the correlation information between the factors.

The technique developed in this paper can be used on active vibration control, fault diagnosis, and condition monitoring for rotating machinery. The application of this method will be reported in the future.

Appendix A. Nomenclature

$Cuvw$	body-fixed co-ordinate system
$Cxyz$	intermediate non-spinning co-ordinate system between $OXYZ$ and $Cuvw$
$OXYZ$	stationary inertial co-ordinate system
h	coefficient of the non-linear structural damping
k	average of the stiffness of the shaft in two directions
k_{ij}	coefficient of the non-linear spring restoring force. $i,j = 0\ldots3$ and $i + j = 3$.
I_t, I_p	diametric and polar moment of inertia of the disk.
$\theta_x,\ \theta_y,\ \dot{\theta}_x,\ \dot{\theta}_y,\ \ddot{\theta}_x,\ \ddot{\theta}_y$	inclination angle, angular velocity, angular acceleration of the disk
$\phi,\ \dot{\phi},\ \ddot{\phi}$	spinning angle, velocity, and acceleration of the rotor
τ	dynamic unbalance
Δ	difference of the stiffness of the shaft in two directions

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