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Metrics on Motion and Deformation of Solid Models

Recently, the importance of metrics on the group of rigid body motions has been addressed in a number of works in the kinematics and robotics literature. This paper defines new metrics on motion which are particularly easy to compute. It is shown how these metrics are applicable to path generation for rigid body motions, and also as a means for generating interpolated sequences of deformed solid models. In order to address both problems in a unified framework, general metrics on Lie groups are discussed.

1 Introduction

Means of measuring "distance" between motions/deformations of solid models are presented. All of the distance functions presented here satisfy the definition of a metric:

Definition: Given a set X, a *metric* is a real-valued function, $d : X \times X \to \mathbb{R}$, which has the following properties for all x_i, x_j, $x_k \in X$:

$$d(x_i, x_j) \geq 0 \quad \text{and} \quad d(x_i, x_j) = 0 \quad \text{iff} \quad x_i = x_j$$

$$d(x_i, x_j) = d(x_j, x_i)$$

$$d(x_i, x_j) + d(x_j, x_k) \geq d(x_i, x_k). \tag{1}$$

We refer to these as positive definiteness, symmetry, and the triangle inequality, respectively. The pair (X, d) is called a *metric space*.

The problem of measuring distance between rigid body motions arises in several scenarios. In mechanism synthesis problems typically a set of frames must be reached, and error measured between actual and desired frames. In computer graphics, both rigid-body and deformable-body motion interpolation are important problems. Both error calculation and motion interpolation require measures of distances to be defined.

A number of recent works have addressed the problem of metrics on the group of rigid body motions. For instance, the exponential/logarithm mappings are used in [6] for an elegant formulation of a physically meaningful metric. In a number of works, metrics based on matrix norms are presented [10, 11]. Yet another approach is presented in [9], in which distances between corresponding points in a rigid body in two different poses are measured and added together to get a measure of distance between the poses. Most recently, interpolation of Euclidean motions has been performed by a procedure analogous to stereographic projection [13].

We introduce new metrics on the set of rigid body motions which are related to those derived previously, but which also extend naturally to more general matrix Lie groups which act on Euclidean space. This is important because deformations of this sort arise in computer vision and graphics (e.g., affine transformations) and in image analysis and pattern recognition/matching. By defining metrics on sets of deformations, a means of limiting the scope of searches in recognition problems is established. Also "morphing" procedures can be based on these metrics because interpolated sequences of solid models can be generated which are "equidistant" from each other, as measured in the metrics we define.

This paper is organized as follows. Section 2 derives metrics on certain groups of transformations, including the group of rigid body motions. Section 3 provides numerous explicit examples. Sections 4 applies the results of Sections 2 and 3 to motion and deformation interpolation and path reparametrization.

2 Metrics on the Affine Group

In this section we present two general methods for generating metrics on subgroups of the proper affine group, which is defined below:

Definition: The *proper affine group* is the set of all pairs of the form $g = (A, \mathbf{b})$ where $A \in \mathbb{R}^{N \times N}$, $\det(A) > 0$, and $\mathbf{b} \in \mathbb{R}^N$. The group law is $g_1 \circ g_2 = (A_1 A_2, A_1 \mathbf{b}_2 + \mathbf{b}_1)$, the identity element is $e = (I, \mathbf{0})$, and this group transforms each point $\mathbf{x} \in \mathbb{R}^N$ as $\mathbf{x} \to g \circ \mathbf{x} = A\mathbf{x} + \mathbf{b}$. This is the only kind of group, and the only kind of group action, considered in this paper.

These groups are important in CAD, graphics, and kinematics because many of the common transformations applied to geometric models are in fact elements of this kind of group. For instance, when A is restricted to be a rotation matrix (i.e., $A \in SO(N)$), the transformation $g \circ \mathbf{x}$ for all $\mathbf{x} \in \mathbb{R}^N$ is a rigid motion of $\mathbb{R}^N$. The group of all rigid body motions is denoted as $SE(N)$.[1]

The basic idea behind the two classes of metrics presented here is to use metrics on $\mathbb{R}^N$ and on function spaces on the groups to induce metrics on the groups themselves. In Sections 2.1 and 2.2 "type 1" and "type 2" metrics are introduced, respectively.

2.1 TYPE 1: Metrics on Groups Induced by Metrics on $\mathbb{R}^N$.

Perhaps the most straightforward way to define metrics on subgroups of the affine group is to take advantage of the well-known metrics on $\mathbb{R}^N$. Namely, if $\rho(\mathbf{x})$ is a continuous real-valued function on $\mathbb{R}^N$ which satisfies the properties

$$0 \leq \rho(\mathbf{x}) < \infty \quad \text{and} \quad 0 < \int_{\mathbb{R}^N} \|\mathbf{x}\|^m \rho(\mathbf{x}) dx_1 \ldots dx_N < \infty,$$

for all finite $m \geq 0$, then

$$d(g_1, g_2) = \int_{\mathbb{R}^N} \|g_1 \circ \mathbf{x} - g_2 \circ \mathbf{x}\| \rho(\mathbf{x}) dx_1 \ldots dx_N$$

is a metric when $\|\cdot\|$ is any norm for vectors in $\mathbb{R}^N$ (in particular, the p-norm is denoted $\|\cdot\|_p$). The fact that this is a metric was observed in [9] for the case when $G = SE(N)$. Because the focus of this paper is to create metrics to compare the amount of deformation or motion of a given solid model, natural choices for $\rho(\mathbf{x})$ are either the mass density of the object, or a function which is equal to one on the object and zero otherwise. How-

[1] $SE(N)$ stands for "Special Euclidean" group of N-dimensional space.

ever, it is also possible to choose a function such as $\rho(\mathbf{x}) = e^{-a^2 \mathbf{x} \cdot \mathbf{x}}$ (for any nonzero $a \in \mathbb{R}$) which is positive everywhere yet decreases rapidly enough for $d(g_1, g_2)$ to be finite.

The fact that this is a metric on the affine transformations which are commonly applied to solid models is observed as follows. The symmetry property $d(g_1, g_2) = d(g_2, g_1)$ results from the symmetry of vector addition and the properties of vector norms. The triangle inequality also follows from properties of norms. Positive definiteness of this metric follows from the fact that for affine transformations

$$\|g_1 \circ \mathbf{x} - g_2 \circ \mathbf{x}\| = \|(A_1 - A_2)\mathbf{x} + (\mathbf{b}_1 - \mathbf{b}_2)\|,$$

and because of the positive definiteness of $\|\cdot\|$, the only time this quantity can be zero is when

$$(A_1 - A_2)\mathbf{x} = \mathbf{b}_2 - \mathbf{b}_1.$$

If $(A_1 - A_2)$ is invertible, this only happens at one point, i.e., $\mathbf{x} = (A_1 - A_2)^{-1}(\mathbf{b}_2 - \mathbf{b}_1)$.

In any case, the set of all $\mathbf{x}$ for which this equation is satisfied will have dimension less than N when $g_1 \neq g_2$, and so the value of the integrand at these points does not contribute to the integral. Thus, because $\rho(\mathbf{x})$ satisfies the properties listed above, and $\|g_1 \circ \mathbf{x} - g_2 \circ \mathbf{x}\| > 0$ for $g_1 \neq g_2$ except on a set of measure zero, the integral in the definition of the metric must satisfy $d(g_1, g_2) > 0$ unless $g_1 = g_2$, in which case $d(g_1, g_1) = 0$.

While this does satisfy the properties of a metric, and could be used for CAD and robot design and path planning problems, it has the significant drawback that the integral of a pth root (or absolute value) must be taken. This means that numerical computations are required. For practical problems, devoting computer power to the computation of the metric detracts significantly from other aspects of the application in which the metric is being used. Therefore, this is not a practical metric. On the other hand, we may modify this approach slightly so as to generate metrics which are calculated in closed form. This yields tremendous computational advantages.

Namely, we observe that

$$d^{(p)}(g_1, g_2) = \sqrt[p]{\int_{\mathbb{R}^N} \|g_1 \circ \mathbf{x} - g_2 \circ \mathbf{x}\|_p^p \rho(\mathbf{x}) dx_1 \dots dx_N}$$

is a metric. Clearly, this is symmetric and positive definite for all of the same reasons as the metric presented earlier. In order to prove the triangle inequality, we must use Minkowski's inequality. That is, if $a_1, a_2, \dots a_n$ and $b_1, b_2, \dots b_n$ are nonnegative real numbers and $p > 1$, then

$$\left(\sum_{k=1}^{n} (a_k + b_k)^p\right)^{1/p} \leq \left(\sum_{k=1}^{n} a_k^p\right)^{1/p} + \left(\sum_{k=1}^{n} b_k^p\right)^{1/p}. \quad (2)$$

In our case, $a = \|g_1 \circ \mathbf{x} - g_2 \circ \mathbf{x}\|_p [\rho(\mathbf{x})]^{1/p}$, $b = \|g_2 \circ \mathbf{x} - g_3 \circ \mathbf{x}\|_p [\rho(\mathbf{x})]^{1/p}$, summation is replaced by integration, and because $\|g_1 \circ \mathbf{x} - g_2 \circ \mathbf{x}\|_p + \|g_2 \circ \mathbf{x} - g_3 \circ \mathbf{x}\|_p \geq \|g_1 \circ \mathbf{x} - g_3 \circ \mathbf{x}\|_p$, then

$$d^{(p)}(g_1, g_2) + d^{(p)}(g_2, g_3)$$

$$\geq \sqrt[p]{\int_{\mathbb{R}^N} (\|g_1 \circ \mathbf{x} - g_2 \circ \mathbf{x}\|_p + \|g_2 \circ \mathbf{x} - g_3 \circ \mathbf{x}\|_p)^p \rho(\mathbf{x}) dx_1 \dots dx_N}$$

$$\geq \sqrt[p]{\int_{\mathbb{R}^N} \|g_1 \circ \mathbf{x} - g_3 \circ \mathbf{x}\|_p^p \rho(\mathbf{x}) dx_1 \dots dx_N}$$

$$= d^{(p)}(g_1, g_3).$$

From the above argument, we can conclude that $d^{(p)}(\cdot, \cdot)$ is a metric. Likewise, it is easy to see that

$$d^{(p')}(g_1, g_2) = \sqrt[p]{\frac{\int_{\mathbb{R}^N} \|g_1 \circ \mathbf{x} - g_2 \circ \mathbf{x}\|_p^p \rho(\mathbf{x}) dx_1 \dots dx_N}{\int_{\mathbb{R}^N} \rho(\mathbf{x}) dx_1 \dots dx_N}}$$

is also a metric, since division by a positive real constant has no effect on metric properties.

The obvious benefit of using $d^{(p)}(\cdot, \cdot)$ or $d^{(p')}(\cdot, \cdot)$ is that the pth root is now outside of the integral, and so the integral can be calculated in closed form.

A particularly useful case is when $p = 2$. In this case it is easy to see that all of the metrics presented in this section satisfy the property

$$d(h \circ g_1, h \circ g_2) = d(g_1, g_2)$$

where g_1 and g_2 are arbitrary affine transformations and $h \in SE(N)$ is a rigid body motion. This is because if $h = (R, \mathbf{b}) \in SE(N)$, then

$$\|h \circ g_1 \circ \mathbf{x} - h \circ g_2 \circ \mathbf{x}\|_2 = \|R[g_1 \circ \mathbf{x}] + \mathbf{b} - R[g_2 \circ \mathbf{x}] - \mathbf{b}\|_2$$

$$= \|g_1 \circ \mathbf{x} - g_2 \circ \mathbf{x}\|_2$$

It is also interesting to note that there is a relationship between the type 1 metric for $SE(N)$ and the Hilbert-Schmidt norm of $N \times N$ matrices. That is, for $g \in SE(N)$

$$d^{(2)}(g, e) = \sqrt{\int_V \|g \circ \mathbf{x} - \mathbf{x}\|_2^2 \rho(\mathbf{x}) dV}$$

is the same as a weighted norm

$$\|g - e\|_W = \sqrt{\text{tr} ((g - e)^T W (g - e))},$$

where $W = W^T \in \mathbb{R}^{4 \times 4}$ and g and e are expressed as 4×4 homogeneous transformations matrices. To show that this is true, we begin our argument by expanding as follows (let $g = (R, \mathbf{b})$):

$$\|g \circ \mathbf{x} - \mathbf{x}\|_2^2 = (R\mathbf{x} + \mathbf{b} - \mathbf{x})^T (R\mathbf{x} + \mathbf{b} - \mathbf{x})$$

$$= \text{tr} ((R\mathbf{x} + \mathbf{b} - \mathbf{x})(R\mathbf{x} + \mathbf{b} - \mathbf{x})^T)$$

$$= \text{tr} ((R\mathbf{x} + \mathbf{b} - \mathbf{x})(\mathbf{x}^T R^T + \mathbf{b}^T - \mathbf{x}^T))$$

$$= \text{tr} (R^T \mathbf{x} \mathbf{x}^T R + R\mathbf{x} \mathbf{b}^T - R\mathbf{x} \mathbf{x}^T + \mathbf{b} \mathbf{x}^T R^T + \mathbf{b} \mathbf{b}^T$$
$$- \mathbf{b} \mathbf{x}^T - \mathbf{x} \mathbf{x}^T R^T - \mathbf{x} \mathbf{b}^T + \mathbf{x} \mathbf{x}^T)$$

$$= \text{tr} (\mathbf{x} \mathbf{x}^T + R\mathbf{x} \mathbf{b}^T - R\mathbf{x} \mathbf{x}^T + R\mathbf{x} \mathbf{b}^T + \mathbf{b} \mathbf{b}^T$$
$$- \mathbf{x} \mathbf{b}^T - R\mathbf{x} \mathbf{x}^T - \mathbf{x} \mathbf{b}^T + \mathbf{x} \mathbf{x}^T)$$

$$= 2 \text{tr} (\mathbf{x} \mathbf{x}^T + R\mathbf{x} \mathbf{b}^T - R\mathbf{x} \mathbf{x}^T - \mathbf{x} \mathbf{b}^T + \tfrac{1}{2} \mathbf{b} \mathbf{b}^T)$$

$$= 2 \text{tr} [(I - R)\mathbf{x} \mathbf{x}^T + \tfrac{1}{2} \mathbf{b} \mathbf{b}^T + R\mathbf{x} \mathbf{b}^T - \mathbf{x} \mathbf{b}^T].$$

I is the 3×3 identity matrix. Note that

$$\int_V \text{tr} [F(\mathbf{x})] \rho(\mathbf{x}) dV = \text{tr} \left[\int_V F(\mathbf{x}) \rho(\mathbf{x}) dV\right],$$

where $F(\mathbf{x})$ is any matrix function, and if we choose the coordinate system at the center of mass, we get $\int_V \mathbf{x} \rho(\mathbf{x}) dV = \mathbf{0}$, and so the type 1 metric is:

$$(d^{(2)}(g, e))^2 = 2 \text{tr} \left[(I - R) \int_V \mathbf{x} \mathbf{x}^T \rho(\mathbf{x}) dV\right]$$

$$+ \mathbf{b} \cdot \mathbf{b} \int_V \rho(\mathbf{x}) dV$$

or

$$d^{(2)}(g, e) = \sqrt{2 \, \text{tr} \, [(I - R)J] + \mathbf{b} \cdot \mathbf{b} M}. \qquad (3)$$

$M = \int_V \rho(\mathbf{x})dV$ is the mass and $J = \int_V \mathbf{x}\mathbf{x}^T \rho(\mathbf{x})dV$ has a simple relationship with the moment of inertia matrix of the rigid body:

$$I_{\text{inertia}} = \int_{\text{vol}} (\mathbf{x}^T \mathbf{x} I - \mathbf{x}\mathbf{x}^T)\rho(\mathbf{x})dV = \text{tr} \, (J)I - J.$$

Now we compare this to the weighted norm of $\|g - e\|_W$, defined by $\|g - e\|_W^2 = \text{tr} \, ((g - e)W(g - e)^T)$, where $W = \begin{pmatrix} W_{3\times3} & 0 \\ 0^T & w_{44} \end{pmatrix}$:

$$\|g - e\|_W^2 = \text{tr} \, ((g - e)W(g - e)^T)$$

$$= \text{tr} \, ((g - e)^T(g - e)W)$$

$$= \text{tr} \left(\begin{pmatrix} R^T - I & 0 \\ \mathbf{b}^T & 0 \end{pmatrix} \begin{pmatrix} R - I & \mathbf{b} \\ 0^T & 0 \end{pmatrix} \begin{pmatrix} W_{3\times3} & 0 \\ 0^T & w_{44} \end{pmatrix} \right)$$

$$= \text{tr} \left(\begin{pmatrix} (R^T - I)(R - I) & (R^T - I)\mathbf{b} \\ \mathbf{b}^T(R - I) & \mathbf{b}^T\mathbf{b} \end{pmatrix} \right.$$

$$\left. \times \begin{pmatrix} W_{3\times3} & 0 \\ 0^T & w_{44} \end{pmatrix} \right)$$

$$= \text{tr} \begin{pmatrix} (R^T - I)(R - I)W_{3\times3} & (R^T - I)\mathbf{b}w_{44} \\ \mathbf{b}^T(R - I)W_{3\times3} & \mathbf{b}^T\mathbf{b}w_{44} \end{pmatrix}$$

$$= \text{tr} \, ((R^T - I)(R - I)W_{3\times3}) + w_{44}\mathbf{b}^T\mathbf{b}$$

$$= 2 \, \text{tr} \, ((I - R)W_{3\times3}) + w_{44}\mathbf{b}^T\mathbf{b}.$$

It is exactly in the same form as Eq. (3), with $J = W_{3\times3}$ and $M = w_{44}$. Therefore, we conclude that

$$d^{(2)}(g, e) = \|g - e\|_W.$$

Furthermore, we can prove that $d(g_1, g_2) = \|g_1 - g_2\|_W$. First note that

$$g_1 - g_2 = \begin{pmatrix} R_1 - R_2 & \mathbf{b}_1 - \mathbf{b}_2 \\ 0^T & 0 \end{pmatrix},$$

so

$$\|g_1 - g_2\|_W^2$$

$$= \text{tr} \left(\begin{pmatrix} R_1^T - R_2^T & 0 \\ \mathbf{b}_1^T - \mathbf{b}_2^T & 0 \end{pmatrix} \begin{pmatrix} R_1 - R_2 & \mathbf{b}_1 - \mathbf{b}_2 \\ 0^T & 0 \end{pmatrix} \begin{pmatrix} W_{3\times3} & 0 \\ 0^T & w_{44} \end{pmatrix} \right)$$

$$= \text{tr} \left(\begin{pmatrix} (R_1^T - R_2^T)(R_1 - R_2) & (R_1^T - R_2^T)(\mathbf{b}_1 - \mathbf{b}_2) \\ (\mathbf{b}_1^T - \mathbf{b}_2^T)(R_1 - R_2) & (\mathbf{b}_1 - \mathbf{b}_2)(\mathbf{b}_1 - \mathbf{b}_2) \end{pmatrix} \right.$$

$$\left. \times \begin{pmatrix} W_{3\times3} & 0 \\ 0^T & w_{44} \end{pmatrix} \right)$$

$$= \text{tr} \, ((R_1^T - R_2^T)(R_1 - R_2)W_{3\times3}) + w_{44}(\mathbf{b}_1 - \mathbf{b}_2)^T(\mathbf{b}_1 - \mathbf{b}_2)$$

$$= \text{tr} \, ((2I - R_1^T R_2 - R_2^T R_1)W_{3\times3}) + w_{44}(\mathbf{b}_1 - \mathbf{b}_2)^T(\mathbf{b}_1 - \mathbf{b}_2).$$

Note that:

$$\text{tr} \, (R_2^T R_1 W_{3\times3}) = \text{tr} \, (W_{3\times3} R_1^T R_2) = \text{tr} \, (R_1^T R_2 W_{3\times3}).$$

We get:

$$\|g_1 - g_2\|_W^2 = 2 \, \text{tr} \, ((I - R_1^T R_2)W_{3\times3}) + w_{44}\|\mathbf{b}_1 - \mathbf{b}_2\|^2.$$

On the other hand, we already know that:

$$d(g_2^{-1} \circ g_1, e) = d(g_1^{-1} \circ g_2, e) = d(g_1, g_2).$$

Noting that:

$$g_1^{-1} \circ g_2 = \begin{pmatrix} R_1^T & -R_1^T \mathbf{b}_1 \\ 0^T & 1 \end{pmatrix} \begin{pmatrix} R_2 & \mathbf{b}_2 \\ 0^T & 1 \end{pmatrix}$$

$$= \begin{pmatrix} R_1^T R_2 & R_1^T(\mathbf{b}_2 - \mathbf{b}_1) \\ 0^T & 1 \end{pmatrix},$$

we get:

$$(d^{(2)}(g_1^{-1} \circ g_2, e))^2$$

$$= \|g_1^{-1} \circ g_2 - e\|_W^2$$

$$= 2 \, \text{tr} \, ((I - R_1^T R_2)W_{3\times3} + w_{44}\|R_1^T(\mathbf{b}_2 - \mathbf{b}_1)\|^2$$

$$= 2 \, \text{tr} \, ((I - R_1^T R_2)W_{3\times3} + w_{44}\|(\mathbf{b}_2 - \mathbf{b}_1)\|^2.$$

This is exactly identical to $\|g_1 - g_2\|_W^2$. So we may conclude that for $g_1, g_2 \in SE(N)$

$$d^{(2)}(g_1, g_2) = \|g_1 - g_2\|_W, \qquad (4)$$

where

$$W = \begin{pmatrix} J & 0 \\ 0^T & M \end{pmatrix}.$$

This property of type 1 metrics is very convenient since we can use many well-developed theories of matrix norms, and the only explicit integration that is required to compute the metric is the computation of moments of inertia (which are already tabulated for most common engineering shapes).

2.2 TYPE 2: Metrics on Groups Induced by Metrics on Their Function Spaces.

Given an arbitrary Lie group, it is always possible to define a piecewise-continuous real-valued function $f: G \to \mathbb{R}$. Furthermore, it is possible to integrate such a function over the group provided $f(g)$ decays rapidly enough. In this case the measure of the function,

$$\mu(f) = \int_G f(g)d\mu(g),$$

is finite where $d\mu(g)$ is an integration measure on the group [15, 14]. On every Lie group, one can define integration measures $d\mu_L(g)$ and $d\mu_R(g)$ such that

$$\mu_L(f) = \int_G f(h \circ g)d\mu_L(g) = \int_G f(g)d\mu_L(g)$$

and

$$\mu_R(f) = \int_G f(g \circ h)d\mu_R(g) = \int_G f(g)d\mu_R(g)$$

for fixed $h \in G$. $\mu_L(f)$ is called left-invariant and $\mu_R(f)$ is called right-invariant. Usually, $\mu_L \neq \mu_R$. In cases when left and right invariant measures are the same, we simply denote it as μ. Any group which has a bi-invariant (left-right invariant) measure is called unimodular. This is important because the group of rigid body motions, $SE(N)$, is just such a group [7].

Given this background, it is possible to define left or right invariant metrics on arbitrary Lie groups in the following way: Let $f(g)$ be a continuous nonperiodic p-integrable function (i.e., $\mu(|f|^p)$ is finite for $\mu = \mu_R$ or $\mu = \mu_L$). Then the following are metrics:

$$d_L^{(p)}(g_1, g_2) = \left(\int_G |f(g_1^{-1} \circ g) - f(g_2^{-1} \circ g)|^p d\mu_L(g) \right)^{1/p},$$

$$d_R^{(p)}(g_1, g_2) = \left(\int_G |f(g \circ g_1) - f(g \circ g_2)|^p d\mu_R(g) \right)^{1/p}.$$

The fact that these are metrics follow easily. The triangle inequality holds from the Minkowski inequality, symmetry holds from the symmetry of scalar addition, and positive definiteness follows from the fact that we choose $f(g)$ to be a nonperiodic continuous function. That is, we choose $f(g)$ such that the equalities $f(g) = f(g_1 \circ g)$ and $f(g) = f(g \circ g_1)$ do not hold for $g_1 \neq e$ except on sets of measure zero. Thus, there is no way for the integral of the difference of shifted versions of this function to be zero other than when $g_1 = g_2$, where it must be zero.

We prove the left-invariance of $d_L^{(p)}(g_1, g_2)$ below. The proof for right-invariance for $d_R^{(p)}(g_1, g_2)$ follows analogously.

$$d_L^{(p)}(h \circ g_1, h \circ g_2)$$

$$= \left(\int_G |f((h \circ g_1)^{-1} \circ g) - f((h \circ g_2)^{-1} \circ g)|^p d\mu_L(g) \right)^{1/p}$$

$$= \left(\int_G |f((g_1^{-1} \circ h^{-1}) \circ g) - f((g_2^{-1} \circ h^{-1}) \circ g)|^p d\mu_L(g) \right)^{1/p}$$

$$= \left(\int_G |f(g_1^{-1} \circ (h^{-1} \circ g)) - f(g_2^{-1} \circ (h^{-1} \circ g))|^p d\mu_L(g) \right)^{1/p}.$$

Because of the left-invariance of μ_L, we then have

$$= \left(\int_G |f(g_1^{-1} \circ g') - f(g_2^{-1} \circ g')|^p d\mu_L(h \circ g') \right)^{1/p}$$

$$= d_L^{(p)}(g_1, g_2),$$

where the change of variables $g' = h^{-1} \circ g$ has been made.

A class function on a group G is a function with the property $f(g \circ h) = f(h \circ g)$ for $g, h \in G$. If a continuous p-integrable class function, f, exists and G is unimodular, then it is always possible to define a bi-invariant metric on G. This is clear as follows, by illustrating the right invariance of a left-invariant metric.

$$d_L^{(p)}(g_1 \circ h, g_2 \circ h)$$

$$= \left(\int_G |f((g_1 \circ h)^{-1} \circ g) - f((g_2 \circ h)^{-1} \circ g)|^p d\mu(g) \right)^{1/p}$$

$$= \left(\int_G |f(h^{-1} \circ g_1^{-1} \circ g) - f(h^{-1} \circ g_2^{-1} \circ g)|^p d\mu(g) \right)^{1/p}$$

For a class function $f(g_1 \circ (g_2 \circ g_3)) = f((g_2 \circ g_3) \circ g_1)$ for any $g_1, g_2, g_3 \in G$, and so

$$d_L^{(p)}(g_1 \circ h, g_2 \circ h)$$

$$= \left(\int_G |f(g_1^{-1} \circ g \circ h^{-1}) - f(g_2^{-1} \circ g \circ h^{-1})|^p d\mu(g) \right)^{1/p}$$

$$= \left(\int_G |f(g_1^{-1} \circ g') - f(g_2^{-1} \circ g')|^p d\mu(g' \circ h) \right)^{1/p}$$

$$= d_L^{(p)}(g_1, g_2),$$

where the substitution $g' = g \circ h^{-1}$ has been made and the right invariance of the integration has been assumed.

It is worth noting that all compact Lie groups are unimodular, and it is always possible to define continuous square-integrable class functions on a compact group. Therefore, by the construction above, it is always possible to define bi-invariant metrics on compact groups. On the other hand, there are no nontrivial square integrable class functions on $SE(N)$, as proved in [16], and so this construction cannot be used to generate bi-invariant metrics on $SE(N)$ for $p \geq 2$. This is consistent with results reported in the literature [6]. However, it does not rule out the existence of anomalous metrics such as the trivial metric, which can be generated for $p = 1$ when $f(g)$ is a delta function on $SE(N)$.

As a practical matter, $p = 1$ is a difficult case to work with since the integration must be performed numerically. Likewise, $p > 2$ does not offer computational benefits, and so we concentrate on the case $p = 2$. In this case we get:

$$d_L^{(2)}(g_1, g_2) = \sqrt{\frac{1}{2} \int_G |f(g_1^{-1} \circ g) - f(g_2^{-1} \circ g)|^2 d\mu(g)}.$$

In the above equation, and henceforth throughout the paper, f is assumed not to be a class function. Note that introducing the factor of $\frac{1}{2}$ does not change the fact that this is a metric. Expanding the square, we see

$$2(d_L^{(2)}(g_1, g_2))^2 = \int_G f^2(g_1^{-1} \circ g) d\mu(g) + \int_G f^2(g_2^{-1} \circ g) d\mu(g)$$

$$- 2 \int_G f(g_1^{-1} \circ g) f(g_2^{-1} \circ g) d\mu(g). \quad (5)$$

Because of the left invariance of the measure, the first two integrals are equal. Furthermore, if we define $f^*(g) = f(g^{-1})$ then the last term may be written as a convolution. That is,

$$d_L^{(2)}(g_1, g_2) = \sqrt{\|f\|_2^2 - (f * f^*)(g_1^{-1} \circ g_2)}, \quad (6)$$

where

$$\|f\|_2^2 = \int_G f^2(g) d\mu(g).$$

In general the convolution of functions on unimodular Lie groups is defined as [14]

$$(\alpha * \beta)(g) = \int_G \alpha(h) \beta(h^{-1} \circ g) d\mu(h).$$

This is a straight forward extension of the definition of convolution of functions on the real line, which is of the form

$$(\alpha * \beta)(y) = \int_R \alpha(x) \beta(-x + y) dx.$$

For the case when $G = SE(N)$ this integral has significance in workspace generation and analysis of discretely actuated manipulators and propagation of kinematic errors in serial linkages [4, 5, 17].

Since the maximum value of $(f * f^*)(g)$ occures at $g = e$ and has the value $\|f\|^2$, we see that $d_L(g_1, g_1) = 0$, as must be the case for it to be a metric. The left invariance is clearly evident when written in the form of Eq. (6), since $(h \circ g_1)^{-1} \circ (h \circ g_2) = g_1^{-1} \circ (h^{-1} \circ h) \circ g_2 = g_1^{-1} \circ g_2$. We also recognize that unlike the class of metrics in the previous section, this one has a bounded value. That is, when we choose $f \geq 0$,

$$\max_{g_1, g_2 \in G} \sqrt{\|f\|_2^2 - (f * f^*)(g_1^{-1} \circ g_2)} \leq \|f\|_2.$$

2.3 Modified Type 2 Metrics. While the type 2 metrics of the previous subsection use integration in their definition, as a practical matter, one would like to avoid explicitly com-

puting integrals. This is achieved by choosing the functions $f(g)$ in such a way that all the integrals have closed-form solutions. One way to guarantee this in the case of compact groups is by using concepts from the *representation theory* of Lie groups.

Definition: A *representation* of a Lie group G is a homomorphism $T : G \to T(G) \subset GL(V)$. V is a vector space called the representation space, and $GL(V)$ is the group of all invertible linear transformations of V onto itself. $T(g)$ for $g \in G$ is expressed in a given basis of V as an invertible matrix, and

$$T(g_1 \circ g_2) = T(g_1)T(g_2) \quad T(g^{-1}) = T^{-1}(g)$$
$$T(e) = I \in GL(V).$$

If the homomorphism is an isomorphism (i.e., $T(g_1) \neq T(g_2)$ implies $g_1 \neq g_2$) then the representation is called *faithful*. Representations that can be expressed as unitary matrices[2] in an orthonormal basis of V are called unitary representations. Such representations play a central role in theoretical physics and harmonic analysis [14].

One observes that

$$D_L^{(2)}(g_1, g_2) = \left(\int_G \|F(g_1^{-1} \circ g) - F(g_2^{-1} \circ g)\|_{HS}^2 d\mu_L(g) \right)^{1/2}$$

is a metric (for exactly the same reasons as $d_L^{(2)}$) when F is an $m \times m$ complex-valued matrix function satisfying analogous conditions to those placed on $f(g)$ in the previous subsection, and $\|\cdot\|_{HS}$ is the Hilbert-Schmidt norm, i.e.,

$$\|F\|_{HS} = \sqrt{\text{tr}\,(FF^*)}$$

where tr $(\cdot)$ denotes the trace of a matrix.

Because of the properties of representation matrices, the above metric can sometimes be easier to compute than $d_L^{(2)}(g_1, g_2)$. For instance, choosing $F = U$ to be an $m \times m$ faithful unitary representation matrix of the compact group G, one uses the properties

$$U(g_1^{-1} \circ g) = U(g_1^{-1})U(g)$$

and

$$\|U(g)\|_{HS} = \sqrt{m}$$

to observe that

$$D_L^{(2')}(g_1, g_2) = \sqrt{2[m - \text{Re}(\text{tr}\,(U(g_1^{-1} \circ g_2)))]}$$

is a metric, where Re$(\cdot)$ denotes the real part of the complex number. This is because all of the g-dependence of the integrand disappears and for compact groups we may always scale the integration measure so that $\int_G d\mu = 1$.

Several closed form examples of type 1 and type 2 metrics are considered in the following section. Specifically, a modification of the approach in this subsection to the case of $SE(N)$ (which is not compact) is presented as an example in Section 3.2.

3 Examples of These Metrics

3.1 Examples of Type 1 Metrics. In the case of rigid body motion of a solid model, we may choose the weighting function $\rho(\mathbf{x})$ to have the physical meaning of the mass density of the solid model. For instance the following metrics on rigid body motion can be generated using ellipses and ellipsoids with uniform mass density.

Example 1: Planar rigid motion of an ellipse:
We select $\rho(\mathbf{x}) = 1$ over the interior of an ellipse and zero otherwise.

Let $g = (R, \mathbf{b}) \in SE(2)$. Then[3]

$$(d^{(2)}(g, e))^2 = \int_A [(r_{11}x_1 + r_{12}x_2 + b_1 - x_1)^2$$
$$+ (r_{21}x_1 + r_{22}x_2 + b_2 - x_2)^2] dx_1 dx_2.$$

Changing to polar coordinates, $(x_1, x_2) = (a\rho \cos\theta, b\rho \sin\theta)$, and performing the integration in this parametrization over $0 \leq \rho \leq 1$ and $0 \leq \theta \leq 2\pi$, a simple calculation yields:

$$d^{(2)}(g, e) = \sqrt{\tfrac{1}{2}[a^2(1 - r_{11}) + b^2(1 - r_{22})] + (b_1^2 + b_2^2)}\,. \tag{7}$$

Example 2: A Nonrigid Example
This approach is not limited to rigid body motions. For instance, if we chose to deform a 2-D solid model by uniformly stretching, shearing, dilating, and rotating (but not translating), then $g \circ \mathbf{x} = A\mathbf{x}$ where A is a general invertible 2×2 matrix with positive determinant. Using the fact that

$$\|\mathbf{x}\|^2 = \mathbf{x} \cdot \mathbf{x} = \text{tr}\,(\mathbf{x}\mathbf{x}^T)$$

it is easy to see that

$$\|(A_1 - A_2)\mathbf{x}\|^2 = \text{tr}\,[(A_1 - A_2)\mathbf{x}\mathbf{x}^T(A_1 - A_2)^T]$$

and thus

$$d^{(2')}(g_1, g_2) = \sqrt{\text{tr}\,[(A_1 - A_2)J(A_1 - A_2)^T]}$$

where

$$J = \frac{1}{\int_{\mathbb{R}^2} \rho(x, y)dxdy} \int_{\mathbb{R}^2} \begin{pmatrix} x^2 & xy \\ yx & y^2 \end{pmatrix} \rho(x, y)dxdy.$$

Thus, no integration is required to compute this metric other than an initial computation of the solid model inertial properties.

Note that in the context of this example

$$d^{(2')}(g_1, g_2) \neq d^{(2')}(e, g_1^{-1} \circ g_2)$$

unless g_1 is a rotation.

3.2 Examples of Metrics of Type 2. In this subsection we consider examples of type 2 metrics on $SE(2)$, $SE(3)$, and $GL(2, \mathbb{R})$ to compare to the examples of the previous subsection.

Example 1: $SE(2)$
In the case of $SE(2)$, homogeneous transforms g and h can be parametrized as

$$g(x, y, \theta) = \begin{pmatrix} \cos\theta & -\sin\theta & x \\ \sin\theta & \cos\theta & y \\ 0 & 0 & 1 \end{pmatrix}$$

and

$$h(\xi, \eta, \alpha) = \begin{pmatrix} \cos\alpha & -\sin\alpha & \xi \\ \sin\alpha & \cos\alpha & \eta \\ 0 & 0 & 1 \end{pmatrix}.$$

We define a parametrized function $f(x, y, \theta)$ by identifying

---

[2] $U^{-1} = U^*$ where * denotes the complex conjugate transpose

[3] We only calculate $d(e, g)$ because by left-invariance $d(g_1, g_2) = d(e, g_1^{-1}g_2)$

$$f(x, y, \theta) = f(g(x, y, \theta)),$$

which leads to an explicit form of the convolution product on $SE(2)$:

$$(f_1 * f_2)(g) = (f_1 * f_2)(x, y, \theta) = \int_{SE(2)} f_1(h) f_2(h^{-1}g) d\mu(h)$$

$$= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\pi}^{\pi} f_1(\xi, \eta, \alpha) f_2((x - \xi)c\alpha + (y - \eta)s\alpha,$$

$$- (x - \xi)s\alpha + (y - \eta)c\alpha, \theta - \alpha) d\xi d\eta d\alpha,$$

where $c\alpha = \cos \alpha$ and $s\alpha = \sin \alpha$. The fact that $dxdyd\theta$ is a bi-invariant integration measure for $SE(2)$ is well known in the literature, e.g., [14].

We choose

$$f(g) = e^{-(x^2+y^2)/2}(1 + \cos \theta),$$

because $f^*(g) = f(g)$ and since it has a single maximum at $(x, y, \theta) = (0, 0, 0)$ it cannot be periodic. Furthermore, we know from previous work ([5]) that the convolution of functions on $SE(2)$ that are products of Hermite functions and trigonometric functions always yield functions of the same form.

For this choice,

$$\|f\|_2^2 = \int_G f^2(g) d\mu(g)$$

$$= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\pi}^{\pi} e^{-(x^2+y^2)}(1 + \cos \theta)^2 dxdyd\theta$$

$$= \int_{-\infty}^{\infty} e^{-x^2} dx \int_{-\infty}^{\infty} e^{-y^2} dy \int_{-\pi}^{\pi} (1 + \cos \theta)^2 d\theta$$

$$= 3\pi^2$$

and

$$(f * f)(g) = \pi^2(2 + \cos \theta)e^{-(x^2+y^2)/4}.$$

The resulting metric is:

$$d_L^{(2)}(g, e) = \pi\sqrt{3 - (2 + \cos \theta)e^{-(x^2+y^2)/4}}. \qquad (8)$$

Example 2: $SE(3)$

Using concepts from Section 2.3 we can generate a modified type 2 metric for $SE(3)$. Note that all the faithful unitary matrix representations of $SE(3)$ are infinite dimensional [14]. This means computing $D_L^{(2)}(g_1, g_2)$ is not practical. However, we can define

$$F(g) = f(R, \mathbf{x}) = e^{-(\mathbf{x} \cdot \mathbf{x}/L^2)} U(R)$$

where $U(\cdot)$ is a faithful unitary representation of $SO(3)$.

These finite dimensional matrices are computed explicitly in [15]. Observing that the bi-invariant volume element for $SE(3)$ is the product of the volume elements for $\mathbb{R}^3$ and $SO(3)$, $d\mu(g) = d\mathbf{x}dR$, direct substitution into the metric presented in Section 2.2 yields

where it is assumed that dR is normalized so that $\int_{SO(3)} dR = 1$, and the unitarity and homomorphism properties of $U(R)$ are used. These properties completely circumvent explicit integration on $SO(3)$. The only remaining integrals are easily computed in closed form. Note that one may simply choose $U(R) = R$.

Example 3: $GL(2, \mathbb{R})$

$GL(2, \mathbb{R})$ is the group of real two-dimensional nonsingular matrices. It can be parametrized as $g = \begin{pmatrix} k & l \\ m & n \end{pmatrix}$, where $k, l, m, n \in \mathbb{R}$. The left-invariant integration measure for this group is [15]

$$d\mu_L(g(k, l, m, n)) = \frac{1}{\det^2(g)} dkdldmdn.$$

Defining the parametrized function as

$$f(g(k, l, m, n)) = e^{-(k^2+l^2+m^2+n^2)} \det(g),$$

we can calculate the type 2 metric on $GL(2, \mathbb{R})$. Note from the discussion in Section 2.2 that $d_L(g_1, g_2)$ is left invariant, so

$$d_L(g_1, g_2) = d_L(g_2^{-1} \circ g_1, g_2^{-1} \circ g_2) = d_L(g_2^{-1} \circ g_1, e). \qquad (9)$$

We let $g_2^{-1} \circ g_1 = h$ and $h^{-1} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$. Therefore, substituting into (5), and performing the integration, we see that

$$2(d_L^{(2)}(h, e))^2$$

$$= 2\int_G f^2(g) d\mu(g) - 2\int_G f(h^{-1} \circ g) f(g) d\mu(g)$$

$$= \frac{\pi^2}{2} - 2\frac{(-(bc) + ad)\pi^2}{1 + a^2 + b^2 + c^2 + b^2c^2 - 2abcd + d^2 + a^2d^2}$$

$$= \frac{\pi^2}{2} - \frac{2\pi^2 \det(h^{-1})}{1 + \|h^{-1}\|_2^2 + \det^2(h^{-1})}.$$

Therefore, the type 2 metric in this case is:

$$d_L^{(2)}(g_1, g_2) = \pi\sqrt{\frac{1}{4} - \frac{\det(g_1^{-1} \circ g_2)}{1 + \|g_1^{-1} \circ g_2\|_2^2 + \det^2(g_1^{-1} \circ g_2)}}. \qquad (10)$$

This contrasts the type 1 metric for the same kind of deformation presented as the second example in the previous subsection.

4 Applications to Motion and Deformation Interpolation

4.1 Path Generation Between Frames. In this section, two-dimensional rigid-body motion path generation is studied to illustrate the applications of type 1 metrics. The problem can be stated as follows:

Given an initial frame $g_0 = e$ (because our metric is $SE(N)$-invariant, we can select the identity homogeneous transformation matrix without loss of generality) and a goal frame $g_n = g$, how can we generate intermediate frames $g_1, g_2, \ldots, g_{n-1}$ which minimize $\sum_{i=0}^{n-1} d(g_i, g_{i+1})$? This problem can be addressed by recursively bisecting two frames in such a way that the total distance is minimized. That is, given a homogeneous transformation g, find $g_{1/2}$, such that $d(e, g_{1/2}) = d(g_{1/2}, g)$ and $d(e,$

$$D_L^{(2)}(e, g_1) = \sqrt{2 \dim(U(R)) \cdot \int_{\mathbb{R}^3} e^{-(2\|\mathbf{x}\|^2)/L^2} d\mathbf{x} - \mathrm{tr}(U(R_1) + U^*(R_1)) \int_{\mathbb{R}^3} e^{-(\|\mathbf{x}\|^2 + \|\mathbf{x} - \mathbf{x}_1\|^2)/(L^2)} d\mathbf{x}}$$

$g_{1/2}) + d(g_{1/2}, g)$ is minimized. Repeat this process until the desired number of frames is generated. Each of these steps is a constrained optimization problem. We can use the method of Lagrange Multipliers to solve it. The cost function is:

$$C = d^2(e, g_{1/2}) + \lambda(d^2(e, g_{1/2}) - d^2(g_{1/2}, g)). \quad (11)$$

Let

$$g_{1/2} = \begin{pmatrix} \cos\theta & -\sin\theta & x_1 \\ \sin\theta & \cos\theta & x_2 \\ 0 & 0 & 1 \end{pmatrix}, g = \begin{pmatrix} \cos\alpha & -\sin\alpha & y_1 \\ \sin\alpha & \cos\alpha & y_2 \\ 0 & 0 & 1 \end{pmatrix},$$

and let $d(\cdot, \cdot)$ be the metric in Eq. (7). Setting all the partial derivatives of C with respect to $x_1, x_2, \theta, \lambda$ to zero, one finds the following solution to the constrained optimization problem:

$$\begin{cases} \lambda = -\dfrac{1}{2} \\ \theta = \dfrac{\alpha}{2} \\ x_1 = \dfrac{y_1}{2} \\ x_2 = \dfrac{y_2}{2} \end{cases} \quad (12)$$

This solution shows that if we use linear interpolation on both the rotational and translational parts of the rigid body motion, then we can minimize the type 1 metric distance along the motion. This method for generating interpolated frames is related to other methods recently presented in the literature [8]. Figure 1 shows an example of motion path generation using this method.

4.2 Equal Partitioning of Frame Paths. In many cases in computer graphics and robotics, a frame path is often known, which is parametrized by a single real number $t \in [0, 1]$. In this section, we seek intermediate frames which evenly segment the path. One intuitive method is to display frames at equal increments of t. The result in this case is not satisfactory (see Fig. 2). In this, and following examples, the parametrization of rotation angle is $\theta(t) = (\pi/2)t^2$ and the parametrization of the translation is $x_1(t) = 10t$ and $x_2(t) = 5J_0(10t)$, where $J_0(t)$ is the zeroth order Bessel function.

We can apply our metrics to this problem. Given the left invariant metrics (type 1 and type 2) which have been derived in this paper, we may define arc length of curves in $SE(N)$ in the following natural way:

Fig. 2 Two dimensional path interpolation. In this figure, only linear interpolation in t is used. The result is not satisfactory.

Fig. 3 Curve reparametrization method is used to get the intermediate frames along the path in this figure. The distance metric is type 1 and $a^2 + b^2 = 5$, which means the rotation part does not have much weight. The result is much better than previous ones.
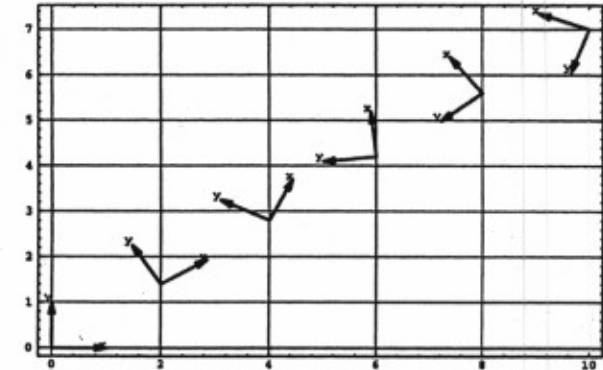

Fig. 1 Two dimensional motion interpolation. The starting frame is the identity and the goal frame is rotated by 160° and translated by 10 units in the x direction and 7 units in the y direction.
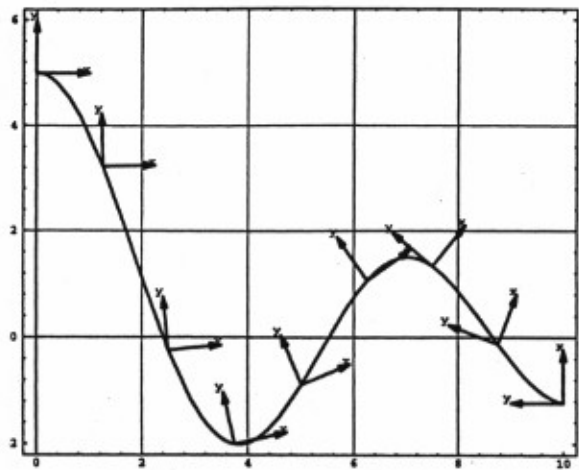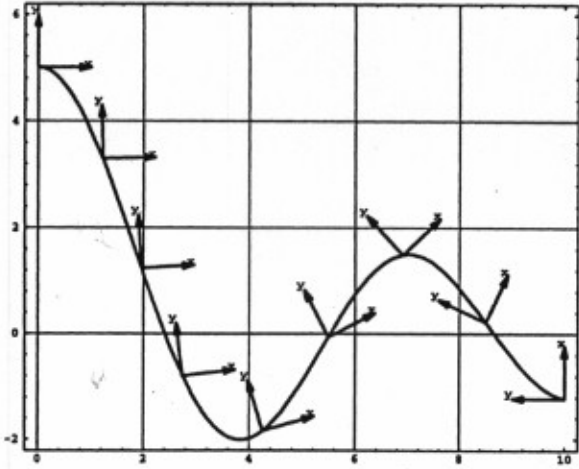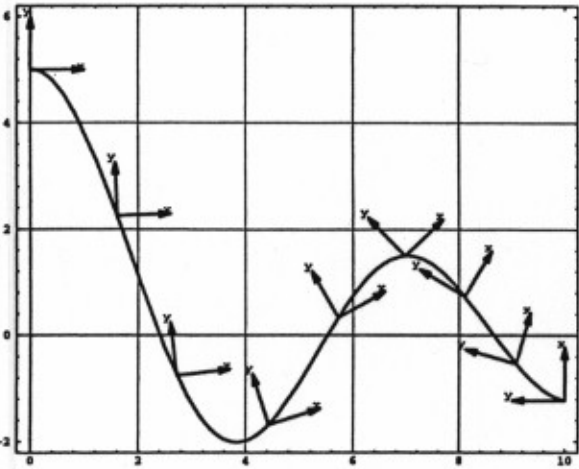

Fig. 4 The difference between this figure and Fig. 3 is that $a^2 + b^2 = 500$ in this figure. We can see clearly that the weighting of the rotation part affects the result of the interpolation.

Given a trajectory $g(t) \in SE(N)$ and given a left-invariant metric, $d(\cdot, \cdot)$, the arc length of $g(t)$ for $t \in [0, 1]$ is

$$L(1) = \lim_{n \to \infty} \sum_{i=0}^{n-1} d\left(g\left(\frac{i}{n}\right), g\left(\frac{i+1}{n}\right)\right) = \lim_{n \to \infty} \sum_{i=0}^{n-1} dL(i). \quad (13)$$

This becomes

$$\begin{aligned} dL &= d(g(t), g(t + dt)) \\ &= d(g(t), g(t) + \dot{g}(t)dt) \\ &= d(e, e + g^{-1}\dot{g}(t)dt) \\ &= f_g(t)dt, \end{aligned}$$

or

$$L(t) = \int_0^t f_g(\sigma)d\sigma,$$

where $f_g(t)$ is a function which depends on the path $g(t)$. We now consider the type 1 metric on $SE(2)$ [Eq. (7)] as a specific example. Let

$$g(t) = \begin{pmatrix} R(t) & \mathbf{x}(t) \\ \mathbf{0}^T & 1 \end{pmatrix},$$

then,

$$e + g^{-1}\dot{g}(t)dt$$

$$= e + \begin{pmatrix} R^T(t) & -R(t)^T\mathbf{x}(t) \\ \mathbf{0}^T & 1 \end{pmatrix}\begin{pmatrix} \dot{R}(t) & \dot{\mathbf{x}}(t) \\ \mathbf{0}^T & 0 \end{pmatrix}dt$$

$$= \begin{pmatrix} I + R^T(t)\dot{R}(t)dt & R^T(t)\dot{\mathbf{x}}(t)dt \\ \mathbf{0}^T & 1 \end{pmatrix},$$

where I is the 2×2 identity matrix. Let θ denote the rotation angle of g, then the rotation part of $e + g^{-1}\dot{g}(t)dt$ is

$$I + R^T(t)\dot{R}(t)dt$$

$$= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}\begin{pmatrix} -\sin\theta & -\cos\theta \\ \cos\theta & -\sin\theta \end{pmatrix}\dot{\theta}dt$$

$$\approx \begin{pmatrix} 1 & -\dot{\theta}dt \\ \dot{\theta}dt & 1 \end{pmatrix}.$$

Let

$$\begin{pmatrix} 1 & -\dot{\theta}dt \\ \dot{\theta}dt & 1 \end{pmatrix} \approx \begin{pmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{pmatrix},$$

where α is an infinitesimal angle. Note $\cos\alpha \approx 1 - (\alpha^2/2)$ and $\sin\alpha \approx 0 + \alpha$, therefore the rotational contribution to $f_g(t)$ when the metric in Eq. (7) is used is

$$\tfrac{1}{2}(a^2(1 - \cos\alpha) + b^2(1 - \cos\alpha)) = \tfrac{1}{4}(a^2 + b^2)\dot{\theta}^2(dt)^2.$$

Because rotation does not change the length of a vector, the translation part is $\|\dot{\mathbf{x}}(t)\|_2^2(dt)^2$. Therefore, for the type 1 metric,
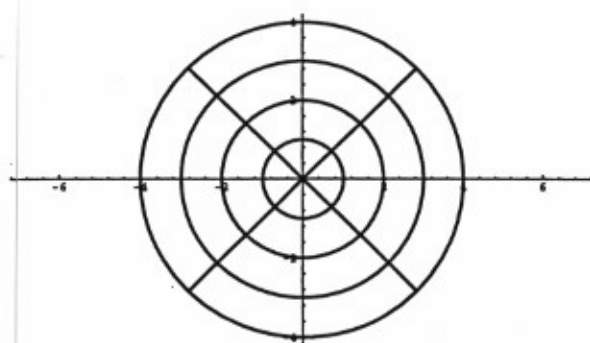

Fig. 5 The initial circle with a radius of 4 length units
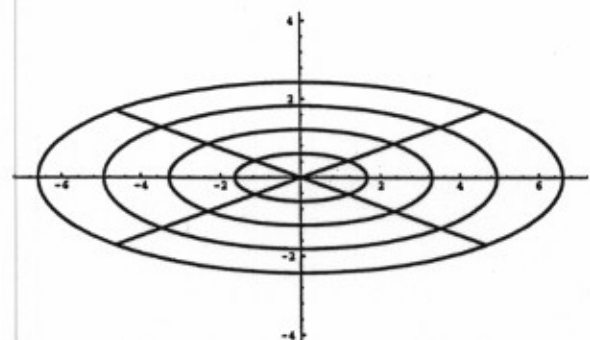
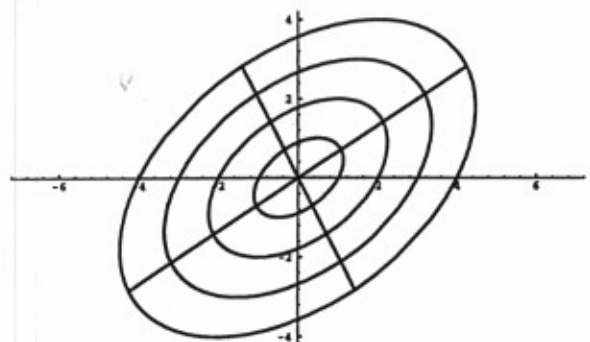

Fig. 6 $SL(2, \mathbb{R})$ transformation of the initial circle with $\theta = 0$, $t = 1$ and $\xi = 0$. The type 1 distance of this transformation from the initial configuration is $d^{(2)}(e, g) = 10.758$ length units^2 or $d^{(2)}(e, g) = 1.517$ length units.

Fig. 7 $SL(2, \mathbb{R})$ transformation of the initial circle with $\theta = 0$, $t = 0$ and $\xi = 0.5$. The type 1 distance of this transformation from the initial configuration is $d^{(2)}(e, g) = 7.0898$ length units^2 or $d^{(2)}(e, g) = 0.9999$ length units.

$$L(t) = \int_0^t \sqrt{\frac{a^2 + b^2}{4}\dot{\theta}^2(\sigma) + \dot{x}_1^2(\sigma) + \dot{x}_2^2(\sigma)}\, d\sigma. \quad (14)$$

Similarly, we can get the corresponding formula for the type 2 metric [Eq. (9)],

$$d^{(2)}(e, e + g^{-1}\dot{g}(t)dt)$$

$$= \pi\sqrt{3 - \left(2 + 1 - \frac{\dot{\theta}^2}{2}(dt)^2\right)\left(1 - \frac{(\dot{x}_1^2 + \dot{x}_2^2)}{4}(dt)^2\right)}$$

$$= \pi\sqrt{\frac{\dot{\theta}^2}{2} + \frac{3(\dot{x}_1^2 + \dot{x}_2^2)}{4}}\, dt.$$
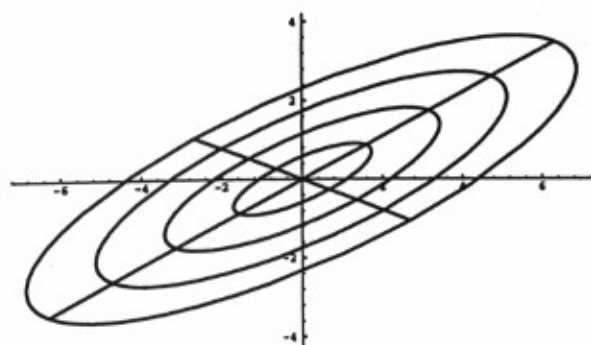
Therefore, for the type 2 metric,

Fig. 8 $SL(2, \mathbb{R})$ transformation of the initial circle with $\theta = -\pi/6$, $t = 1$ and $\xi = 0.5$. The type 1 distance of this transformation from the initial configuration is $d^{(2)}(e, g) = 14.0530$ length units^2 or $d^{(2')}(e, g) = 1.9821$ length units.

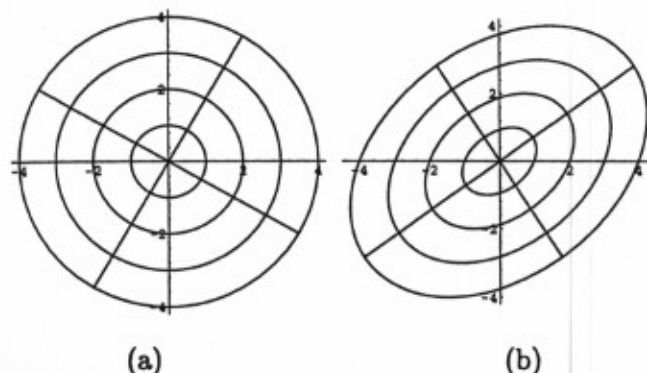(a)   (b)

Fig. 9 (a) is the $SL(2, \mathbb{R})$ rotating transformation of the initial circle Fig. 5 with $\theta = -\pi/6$, $t = 0$, $\xi = 0$. (b) is the $SL(2, \mathbb{R})$ shearing transformation of the same circle with $\theta = 0$, $t = 0$, $\xi = 0.369184$. The distance of these two kinds of transformation from the initial circle are the same under the type 1 metric.

$$L(t) = \pi \int_0^t \sqrt{\frac{\dot{\theta}^2(\sigma)}{2} + \frac{3(\dot{x}_1^2(\sigma) + \dot{x}_2^2(\sigma))}{4}} \, d\sigma. \quad (15)$$

It is interesting to note that arclength calculated in this way can be viewed as a special case of the methods presented in [6, 7], where the choice of a particular solid model (for type 1), or a function on the group (for type 2) fixes constants which are free in that formulation.

Given $L(t)$, one can reparametrize any given path by calculating the total path length, $L(1)$, and increasing the value of t until values $t_1, \ldots, t_N$ are found which satisfy $L(t_i) = (i/N)L(1)$. Figures 3–4 show the result of this method using type 1 and type 2 metrics, respectively. One observes that the result is much better than the original parametrization.

4.3 Applications to Volume Preserving Deformations. Transformations of $\mathbb{R}^2$ by $SL(2, \mathbb{R})$ (the group of real 2×2 matrices with unit determinant) are more general than rigid body rotation. Any matrix in $SL(2, \mathbb{R})$ can be parametrized by rotating (θ), stretching (t) and shearing (ξ) [14]. The composition of these three parameters is:

$$g = \begin{pmatrix} e^{t/2} \cos \dfrac{\theta}{2} & \cos \dfrac{\theta}{2} e^{t/2} \xi + \sin \dfrac{\theta}{2} e^{-t/2} \\ -e^{t/2} \sin \dfrac{\theta}{2} & -\sin \dfrac{\theta}{2} e^{t/2} \xi + \cos \dfrac{\theta}{2} e^{-t/2} \end{pmatrix} \quad (16)$$

Multiplying any element of $SL(2, \mathbb{R})$ by a scale factor of the form e^s for $s \in \mathbb{R}$ produces an element of $GL^+(2, \mathbb{R})$, which is the group of matrices considered in example 3 of Sections 3.1 and 3.2. Alternatively, one may view any element of $SL(2, \mathbb{R})$ as an element of $GL^+(2, \mathbb{R})$, and so the metrics derived earlier can be used in this case.

Figures 6–8 show the transformation of a circle (Fig. 5) with a radius of 4 units under the action of $SL(2, \mathbb{R})$. Using the metrics derived earlier, we can compare different kinds of distortions and motions. Figure 9 shows a rotation and a shear transformation which have the same distance from the identity. Figure 10 shows a shear transformation with equal distance from the identity as a stretch transformation.

Thus, the metrics presented in this paper are a way to quantitatively compare very different kinds of distortions of solid models. While we have limited the scope of the current discussion to affine transformations, type 1 metrics also apply to some other more general deformations of solid models such as those presented in [3].

5 Conclusions

Two classes of metrics on transformation groups of relevance in kinematics and CAD are presented. We focus on explicit examples and properties of metrics on the group of rigid body motions, $SE(N)$. We also show how the same metrics can be used for combinations of motion and deformation of solid models. It is illustrated how metrics derived in this paper can be used to generate interpolated sequences of rigid and/or deformable motions of solid models.
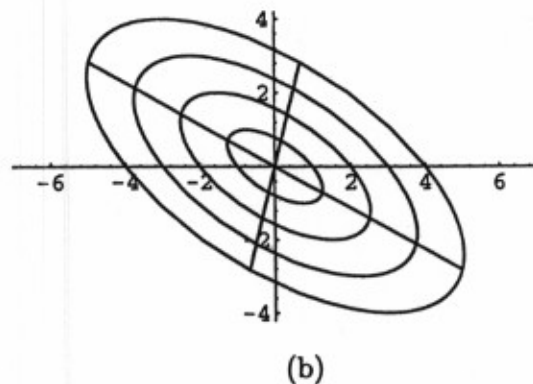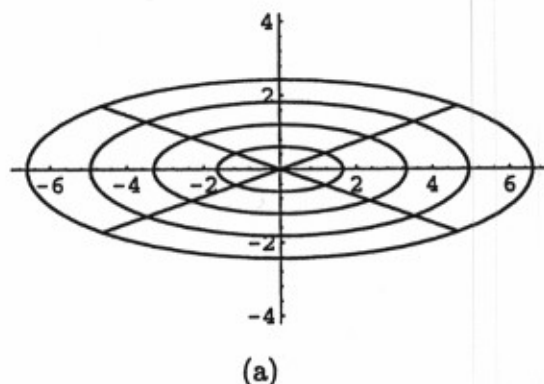

(a)   (b)

Fig. 10 (a) is the $SL(2, \mathbb{R})$ stretching transformation of the initial circle Fig. 5 with $\theta = 0$, $t = 1$, $\xi = 0$. (b) is the $SL(2, \mathbb{R})$ shearing transformation of the same circle with $\theta = 0$, $t = 0$, $\xi = -0.758721$. The distance of these two transformations from the initial circle are the same under the type 1 metric.

References

1 Bröcker, T., tom Dieck, T., 1985, *Representations of Compact Lie Groups*, Springer-Verlag, New York.

2 Sternberg, S., 1994, *Group Theory in Physics*, Cambridge University Press.

3 Chirikjian, G. S., 1995, "Closed-Form Primitives for Generating Locally Volume Preserving Deformations," ASME JOURNAL OF MECHANICAL DESIGN, Sept., Vol. 117, pp. 347–354.

4 Chirikjian, G. S., 1996, "Fredholm Integral Equations on the Euclidean Motion Group," *Inverse Problems 12*, Oct., pp. 579–599.

5 Ebert-Uphoff, I., and Chirikjian, G. S., "Discretely Actuated Manipulator Workspace Generation by Closed-Form Convolution," *Proceedings 1996 ASME Design Engineering Technical Conference and Computers in Engineering Conference*, Irvine, California, August 1996. (12 pages on CD ROM).

6 Park, F. C., 1995, "Distance Metrics on the Rigid-Body Motions with Applications to Mechanism Design," ASME JOURNAL OF MECHANICAL DESIGN, Vol. 117, Mar, 48–54.

7 Park, F. C., and Brockett, R. W., 1994, "Kinematic Dexterity of Robotic Mechanisms," *The International Journal of Robotics Research*, Vol. 13, No. 1, February, pp. 1–15.

8 Park, F. C., and Ravani, B., "Bézier Curves on Riemannian Manifolds and Lie Groups with Kinematics Applications," ASME JOURNAL OF MECHANICAL DESIGN, Vol. 117, Mar., pp. 36–40.

9 Martinez, J. M. R., and Duffy, J., 1995, "On the Metrics of Rigid Body Displacement for Infinite and Finite Bodies," ASME JOURNAL OF MECHANICAL DESIGN, Vol. 117, Mar., pp. 41–47.

10 Kazerounian, K., and Rategar, J., 1997, "Object Norms: A Class of Coordinate and Metric Independent Norms for Displacement," *Flexible Mechanisms, Dynamics, and Analysis*, ASME DE-Vol. 47, pp. 271–275.

11 Fanghella, P., and Galletti, C., 1995, "Metric Relations and Displacement Groups in Mechanism and Robot Kinematic," ASME JOURNAL OF MECHANICAL DESIGN, Vol. 117, Sept., pp. 470–478.

12 Malvern, L. E., 1969, *Introduction to the Mechanics of a Continuous Medium*, Prentice-Hall, Inc., Englewood Cliffs, N.J.

13 Etzel, K. R., and McCarthy, J. M., 1996, "Spatial Motion Interpolation in an Image Space of $SO(4)$," *Proceedings of 1996 ASME Design Engineering Technical Conference and Computers in Engineering Conference*, August 18–22, Irvine, California.

14 Sugiura, M., 1990, *Unitary Representations and Harmonic Analysis*, 2nd edition, North-Holland, Amsterdam.

15 Vilenkin, N. J., and Klimyk, A. U., 1991, *Representation of Lie Groups and Special Functions*, Vols. 1–3, Kluwer Academic Publ., Dordrecht, Holland.

16 Kyatkin, A. B., Chirikjian, G. S., "Regularized Solution of a Nonlinear Convolution Equation on the Euclidean Group," *Acta Appl. Math.*, to appear.

17 Chirikjian, G. S., and Ebert-Uphoff, I., "Numerical Convolution on the Euclidean Group with Applications to Workspace Generation," *IEEE Trans. on Robotics and Automation*, Vol. 14, No. 1, Feb. 1998, pp. 123–136.