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OPTIMAL ONE-PLANE ACTIVE BALANCING OF A RIGID ROTOR DURING ACCELERATION

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1. INTRODUCTION

Rotating machinery, including machining spindles, industrial turbomachinery, and aircraft gas turbine engines, are very commonly used in industry. One major problem faced by these machineries is the harmful, imbalance-induced vibration. Many methods have been developed to reduce this vibration: off-line balancing methods [1], on-line active balancing methods using mass redistribution devices [2–5], and on-line active balancing methods using magnetic bearings [6–9]. These on-line methods can be applied during the operation of the rotor if the rotating speed is a constant.

In some applications, the balancing needs to be completed during speed-varying transient time in order to save time and get better performance. For example, in high-speed machining, the spindle speed could be up to 40 000 r.p.m. and the chip-to-chip time could be less than 2 s. If an active balancing scheme is used in this machine, the balancing has to be done during the acceleration period to avoid increasing the cutting cycle time. Furthermore, the maximum vibration of a rotor usually occurs when it passes through its critical speeds. To avoid this hostile vibration, balancing during acceleration is needed. Zhou and Shi [10] proposed an adaptive active balancing scheme to perform balancing during acceleration by using an innovative mass redistribution actuator. There are multiple vibration modes for a general rotor. In general, if single-plane balancing is considered, the optimal compensating imbalances are different for different modes. However, since the imbalance distribution of the balancer can be changed during operation, the vibration of both vibration modes can be suppressed efficiently by only one balancer [11]. To balance multiple modes with only one balancer, a "switching" function for the balancer needs to be determined. In this paper, we assume the balancer is not at any nodes of the vibration modes and the optimal switching function for the balancer during acceleration is investigated. A rigid rotor model is used, but the extension to the flexible rotor with multiple vibration modes is straightforward.

This paper consists of four sections. A brief review of the adaptive active balancing will be given in section 2. Section 3 presents an optimal one-plane active balancing strategy, which

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is efficient for reducing the imbalance-induced vibration. This strategy is optimal under the given cost function. Finally, conclusions are presented in the last section.

2. REVIEW OF ADAPTIVE ACTIVE BALANCING DURING ACCELERATION

The geometric set-up of a rigid rotor model is shown in Figure 1. XYZ is the inertial co-ordinate frame and xyz is the body-fixed co-ordinate frame. The basic assumptions are the following. (1) The rotor is rigid; the imbalance is modelled as a concentrated mass on the rotor. (2) The bearings are isotropic and modelled by a set of linear spring and dampers. (3) The angular acceleration is constant; the translational motion in Y direction is assumed to be zero. (4) The lateral vibration motions are assumed small to simplify the dynamics.

By denoting $\mathbf{x} = [R_X\ R_Z\ \theta\ \psi\ \dot{R}_X\ \dot{R}_Z\ \dot{\theta}\ \dot{\psi}]^{\mathrm{T}}$ and

$$\mathbf{A}(t) = \begin{bmatrix} 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ -\dfrac{2k}{m} & 0 & 0 & 0 & -\dfrac{2c}{m} & 0 & 0 & 0 \\ 0 & -\dfrac{2k}{m} & 0 & 0 & 0 & -\dfrac{2c}{m} & 0 & 0 \\ 0 & 0 & -\dfrac{kL^2}{2I_t} & \dfrac{I_p\ddot{\phi}}{I_t} & 0 & 0 & -\dfrac{cL^2}{2I_t} & \dfrac{I_p\dot{\phi}}{I_t} \\ 0 & 0 & -\dfrac{I_p\ddot{\phi}}{I_t} & -\dfrac{kL^2}{2I_t} & 0 & 0 & -\dfrac{I_p\dot{\phi}}{I_t} & -\dfrac{cL^2}{2I_t} \end{bmatrix},$$

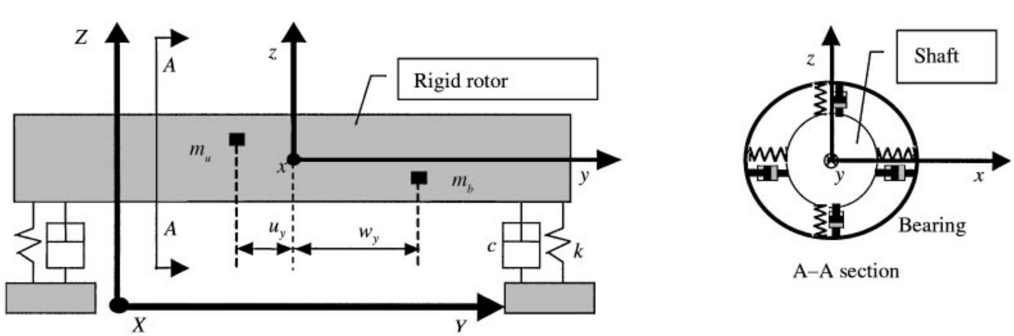

Figure 1. The geometric set-up of the rigid rotor model.

$$
\mathbf{B} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ \dfrac{-m_b w_z - m_u u_z}{m} & \dfrac{m_b w_x + m_u u_x}{m} \\ \dfrac{m_b w_x + m_u u_x}{m} & \dfrac{m_b w_z + m_u u_z}{m} \\ \dfrac{m_b w_x w_y + m_u u_x u_y}{I_t} & \dfrac{m_b w_y w_z + m_u u_y u_z}{I_t} \\ \dfrac{m_b w_y w_z + m_u u_y u_z}{I_t} & \dfrac{-m_b w_x w_y - m_u u_x u_y}{I_t} \end{bmatrix},
$$

we can obtain the governing equation of motion of the rigid rotor during acceleration

$$
\dot{\mathbf{x}} = \mathbf{A}(t)\mathbf{x} + \mathbf{B}\begin{bmatrix} f_1 \\ f_2 \end{bmatrix}, \tag{1}
$$

where $f_1 = \ddot{\phi}\cos\phi - \dot{\phi}^2\sin\phi$, $f_2 = \ddot{\phi}\sin\phi + \dot{\phi}^2\cos\phi$, are known functions of time. The model is a linear time-variant model because $\mathbf{A}(t)$ contains time-variant entries such as $I_p\dot{\phi}(t)/I_t$. However, these time-varying entries are much smaller than other entries in the $\mathbf{A}(t)$ matrix in practice because for a rotor whose diameter is much smaller than its length, I_p is much smaller than I_t. In the following derivation, these time-varying terms are ignored and the system is treated as a time-invariant linear system.

To set up the estimation problem, the continuous model is transformed into a discrete model. If we let $b_1 = m_u u_x/m$, $b_2 = m_u u_z/m$, $b_3 = m_u u_x u_y/m$, $b_4 = m_u u_z u_y/m$ be unknown parameters and $a_1 = k/m$, $a_2 = c/m$, $b_0 = 1/m$, $g_0 = m/I_t$, $g_1 = mL^2/2I_t$, $g_2 = I_p/I_t$ are known parameters, and

$$
y_1 = \frac{\dot{R}_X[(k+1)T] - \dot{R}_X[kT]}{T} - [(m_b w_x f_2 - m_b w_z f_1)b_0 - 2a_2\dot{R}_X - 2a_1 R_X]|_{t=kT},
$$

$$
\mathbf{x}_1 = [f_2 \quad -f_1]^{\mathrm{T}}|_{t=kT} \quad \text{and} \quad \varphi_1 = [b_1 \quad b_2]^{\mathrm{T}},
$$

$$
y_2 = \frac{\dot{\theta}[(k+1)T] - \dot{\theta}|_{t=kT}}{T} - [b_0 m_b w_y g_0(w_x f_1 + w_z f_2) - g_1 a_2\dot{\theta} - g_1 a_1\theta]|_{t=kT},
$$

$$
\mathbf{x}_2 = [g_0 f_1 \;\; g_0 f_2]^{\mathrm{T}}|_{t=kT} \quad \text{and} \quad \varphi_2 = [b_3 \;\; b_4]^{\mathrm{T}},
$$

we can obtain the regression formulation

$$
y_1 = \mathbf{x}_1^{\mathrm{T}}\varphi_1 + \varepsilon_1, \qquad y_2 = \mathbf{x}_2^{\mathrm{T}}\varphi_2 + \varepsilon_2, \tag{2}
$$

where ε_1 and ε_2 are normally distributed noise, which include multiple noise sources, such as the measurement noise and modelling noise. It is well known that the least-squares solution of equation (2) is

$$\hat{\varphi}_1 = (\mathbf{x}_1\mathbf{x}_1^{\mathrm{T}})^{-1}\mathbf{x}_1 y_1, \qquad \hat{\varphi}_2 = (\mathbf{x}_2\mathbf{x}_2^{\mathrm{T}})^{-1}\mathbf{x}_2 y_2. \tag{3}$$

The calculation of equation (3) can be fulfilled in a recursive manner [12]. Based on this solution, b_1–b_4 can be estimated.

The actuator used in this study is a mass redistribution device. If the objective is to minimize the imbalance-induced synchronous vibration, the best way to do this is to let the forcing term be zero. There are two vibration modes for a rigid rotor system. To control the translational mode, we need

$$w_z = -m_u u_z/m_b \quad \text{and} \quad w_x = -m_u u_x/m_b, \tag{4}$$

to offset the imbalance-induced force. To control the conical inclination mode, we need

$$w_z = -m_u u_z u_y/(m_b w_y) \quad \text{and} \quad w_x = -m_u u_x u_y/(m_b w_y), \tag{5}$$

to offset the imbalance-induced moment.

It is clear that if the balancer is not at the same transverse plane as the imbalance, i.e., $w_y \neq u_y$, these two modes cannot be controlled simultaneously by only one balancer. In the next section, an optimal balancing strategy based on the imbalance estimation will be presented.

3. OPTIMAL ACTIVE BALANCING FOR ROTOR SYSTEM WITH MULTIPLE VIBRATION MODES

The imbalance-induced vibration during acceleration needs to be studied in order to obtain the optimal positions of the balancer.

In the past, most work [13] used numerical integration techniques to obtain the transient vibration of a rotor during acceleration. Those techniques are not suitable for the optimal active balancing design. Recently, Zhou and Shi [14] developed an analytical expression to describe the imbalance response of a Jeffcott rotor during acceleration. The imbalance response of the Jeffcott rotor during constant acceleration is written explicitly as a function of the system imbalance. The whole response consists of three parts: (1) transient vibration at damped natural frequency, (2) synchronous vibration at the instantaneous "synchronous" frequency, and (3) suddenly occurring vibration at damped natural frequency. Although the solution is based on the Jeffcott rotor, this solution can be applied to a general rotor by using modal analysis technique. An outline of this solution is given in Appendix A. The part 1 vibration is usually very small. If only parts 2 and 3 are considered, the transient response for a general rotor system can be written as

$$\mathbf{v} = \sum_{k=1}^{N} [\mathbf{U}_k \mathrm{e}^{\mathrm{i}[\frac{\alpha t^2}{2} + \rho_k(t)]} M_{sk}(t) + \mathbf{U}_k \mathrm{e}^{\mathrm{i}[\omega_{dk} t + \gamma_k(t)]} M_{nk}(t)], \tag{6}$$

where $\mathbf{v}$ is a complex number that represents the vibration of rotor in two directions, N is the number of significant vibration modes, $\mathbf{U}_k$ a complex number that represents the system imbalance in the kth mode, t the time, α the acceleration, ω_{dk} the damped natural frequency

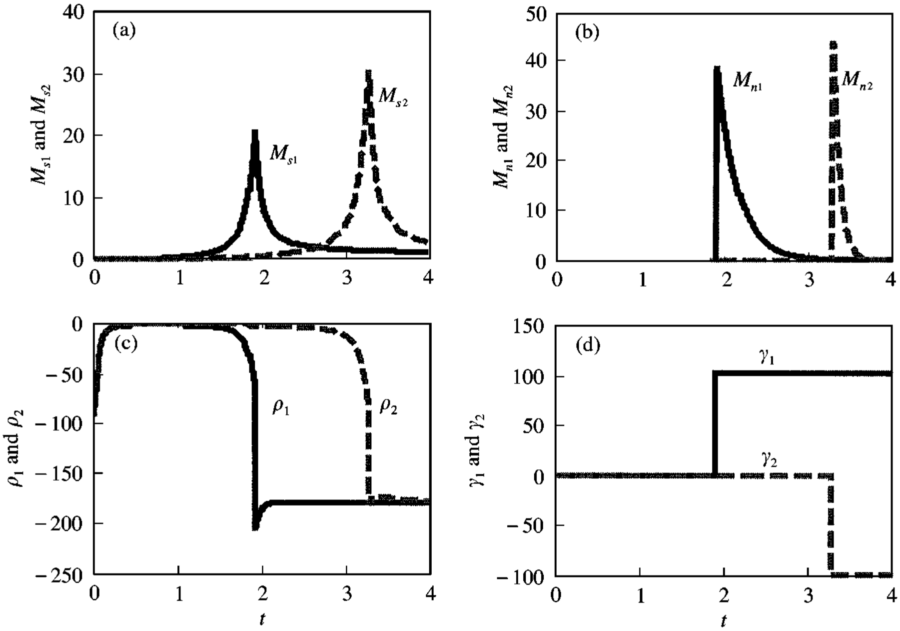

Figure 2. The decomposition of the transient vibration for a rigid rotor: (a) and (c) $M_{sk}(t)$ and $\rho_k(t)$ in equation (6); (c) and (d) $M_{nk}(t)$ and $\gamma_k(t)$ in equation (6).

of the kth mode, and M_{sk}, ρ_k, M_{nk} and γ_k are defined as the magnitudes and phases of the synchronous vibration and the suddenly occurring vibration in the kth mode respectively.

In the rigid rotor cases there are only two vibration modes: translational motion and inclination motion. These two modes are de-coupled in the set-up in Figure 1. Hence, the modal decomposition is not needed. ρ_k, M_{sk}, γ_k, and M_{nk} for a rigid rotor with $\alpha = 300\,\mathrm{rad/s^2}$, $L = 1{\cdot}0$ m, $r = 0{\cdot}1$ m, $\mathrm{k} = 4 \times 10^7$ N/m, and $C = 1000$ N s/m are shown in Figure 2.

A cost function for the optimal active balancing strategy can be

$$c_1 = \min_{\mathbf{u}_{bk}} \left[\sum_{k=1}^{N} \|\mathbf{v}_{sk}\| + \sum_{k=1}^{N} \|\mathbf{v}_{nk}\| \right], \tag{7}$$

where $\mathbf{u}_{bk}$ is the imbalance provided by the balancer for the kth mode, $\mathbf{v}_{sk}$ and $\mathbf{v}_{nk}$ are the synchronous and suddenly occurring vibration for the kth mode. The meaning of this cost function is that we try to minimize the magnitude of the total vibration contributed by N vibration modes during acceleration. The vibration in each mode is represented as a summation of the synchronous component and the suddenly occurring free vibration. Substituting the analytical expression for $\mathbf{v}_{sk}$ and $\mathbf{v}_{nk}$ into equation (7) yields

$$c_1 = \min_{\mathbf{u}_{bk}} \left[\sum_{k=1}^{N} \|(\mathbf{u}_{bk} + \mathbf{u}_{sk})\| \, M_k(t) \right], \tag{8}$$

where $\mathbf{u}_{bk}$ is the imbalance provided by the balancer for the kth mode, $\mathbf{u}_{sk}$ is the system imbalance, and M_k is $M_{sk} + M_{nk}$. One point that needs to be noticed is that the transient vibration caused by the balancer movement is ignored because of the slow movement of the balancer. This assumption will be true only when the balancer movement happens at an instant that is far from the resonant peaks. Indeed, we should not move the balancer at resonant peaks in practice because that will cause large transient vibration even if the balancer movement is slow.

Equation (8) is the cost function for a general rotor system. A rigid rotor with two vibration modes will be used as an example to illustrate the solution procedure for this optimization problem.

For a rigid rotor, defining $a = m_u u_x/m$, $b = m_u u_z/m$, $c = m u_y/I_t$ and $x = m_b w_x/m$, $y = m_b w_z/m$, $d = m w_y/I_t$, we obtain

$$u_{s1} = a + b\mathrm{i},\ u_{s2} = cb - ca\mathrm{i}, \qquad u_{b1} = x + y\mathrm{i},\ u_{b2} = dy - dx\mathrm{i}. \tag{9}$$

Then, equation (8) changes to

$$\min_{x,y} [\sqrt{(a+x)^2 + (b+y)^2} M_1 + \sqrt{(cb+dy)^2 + (ca+dx)^2} M_2]. \tag{10}$$

In this equation, $\sqrt{(a+x)^2 + (b+y)^2} M_1$ represents the translational vibration whose physical unit is meter. $\sqrt{(cb+dy)^2 + (ca+dx)^2} M_2$ represents the conical vibration whose physical unit should originally be radians. However, in order to make these two terms addable, we multiply a scale factor l_s with a unit of meter with the original M_2. Hence, both the first term and the second term have a unit of meter. The physical meaning of this multiplication is that the conical vibration is represented by the vibration of a point that is located at $(0, l_s, 0)$.

The geometrical method can be used to illustrate the solution of this optimization problem. The problem can be re-formulated as follows. Given two points

$$A{:}\ (aM_1, bM_1), \qquad B{:}\ (caM_2, cbM_2), \tag{11}$$

find x, y such that the summation of the distances from A, B to two other points,

$$C{:}\ (-M_1 x,\ -M_1 y), \qquad D{:}\ (-M_2 dx,\ -M_2 dy), \tag{12}$$

reaches minimum.

The solution is shown in Figure 3. It is clear that points O, A, and B are on the same line and points O, C, and D are on the same line from their co-ordinates. Without losing generality, we can assume that OD is longer than OC and that OB is longer than OA.

First, we can prove that the optimal positions of C and D are on the line of OAB. Given arbitrary points C' and D' that are not on the line of OAB, we can find two other points C and D that are on the line of OAB and $|\mathrm{O}C| = |\mathrm{O}C'|$, $|\mathrm{O}D| = |\mathrm{O}D'|$; $|\cdot|$ means length. It is clear that $|\mathrm{O}A| \leqslant |\mathrm{O}C'| + |C'A|$ from the triangle inequality. But $|\mathrm{O}A| = |\mathrm{O}C| + |CA|$, and $|\mathrm{O}C| = |\mathrm{O}C'|$; therefore $|CA| \leqslant |C'A|$. Similarly, $|BD| \leqslant |BD'|$. Hence, the optimal C and D must locate at the line of OAB.

Second, the optimal location of C and D on the line OAB can be found as follows. Since the directions of OAB and OCD are the same, $x/y = a/b$. Substituting $y = (b/a)x$ into equation (10), the minimization problem changes to

$$\min_{x} \sqrt{1 + \frac{b^2}{a^2}} [|a + x| M_1 + |ca + dx| M_2]. \tag{13}$$

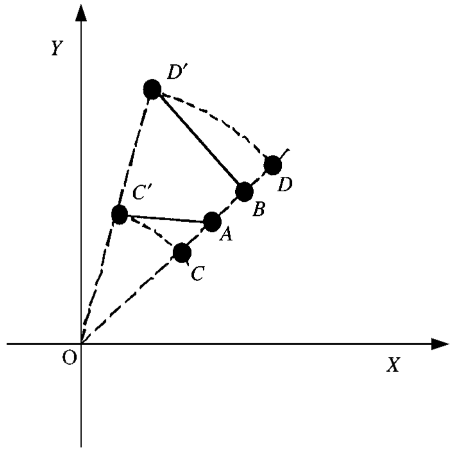

Figure 3. Geometrical solution.

Assume $M_1 > |d|M_2$ and $x = -a$; the cost function is $|ca/d - a||d|M_2$. If $x = -a + q$, where q is an arbitrary number, the cost function is $|q|M_1 + |ca/d - a + q\,||d|M_2$, which is larger or equal to $|q|M_1 + |ca/d - a||\,d|M_2 - |q||d|M_2$. But since $M_1 > |d|M_2$, $|q|M_1 - |qd|M_2 \geqslant 0$. Hence, the cost function, when $x = -a + q$, is larger than the cost function when $x = -a$. Similarly, if $M_1 < |d|M_2$, the solution is $x = -ca/d$.

This optimal solution can be converted to the optimal active balancing strategy. At a certain point when the vibration provided by the second mode multiplied by d becomes larger than the vibration provided by the first mode, the balancer jumps from balancing the first mode to balancing the second mode. The ideal step control policy cannot be realized in practice. We can use the maximum capability of the balancer to approximate this step change.

Figure 4 shows a simulation result using the developed optimal balancing strategy. In this simulation, the imbalance (0·5 kg) is located at (0·1, $-$0·3, 0·05) and the balancer (0·5 kg) is located at (w_x, 0·3, w_z), where w_x and w_z are control inputs. The output is the vibration of a point located at the rotating axis, which is 0·5 m from the center of mass. A noise with range $\pm 10^{-5}$ m is added to the system output. Other dynamic parameters used are the same as those in Figure 2.

Using the optimal active balancing strategy developed, we obtain the switching time for the second mode as $t = 2{\cdot}9506$ s. The optimal active balancing scheme integrates the recursive least-squares estimation of imbalance and the optimal step-changing control law. The optimal balancing strategy adopted by Figure 4(g) is as follows: (1) keep the balancer at zero position and estimate the position and magnitude of the system imbalance; (2) move the balancer to the opposite side of the estimated imbalance to counteract the imbalance in the first vibration mode and keep updating the estimation of system imbalance; (3) move the balancer to control the second vibration mode at 2·9506 s and keep updating the imbalance estimation. (4) balance the second vibration mode after 2·9506 s and update the imbalance estimation.

The movement of the balancer is simplified as a linear function. The duration of the movement is 0·2 s. Comparing the controlled output of the single mode balancing strategy as shown in Figures 4(d) and (f), the optimal control method can reduce the imbalance-induced vibration efficiently.

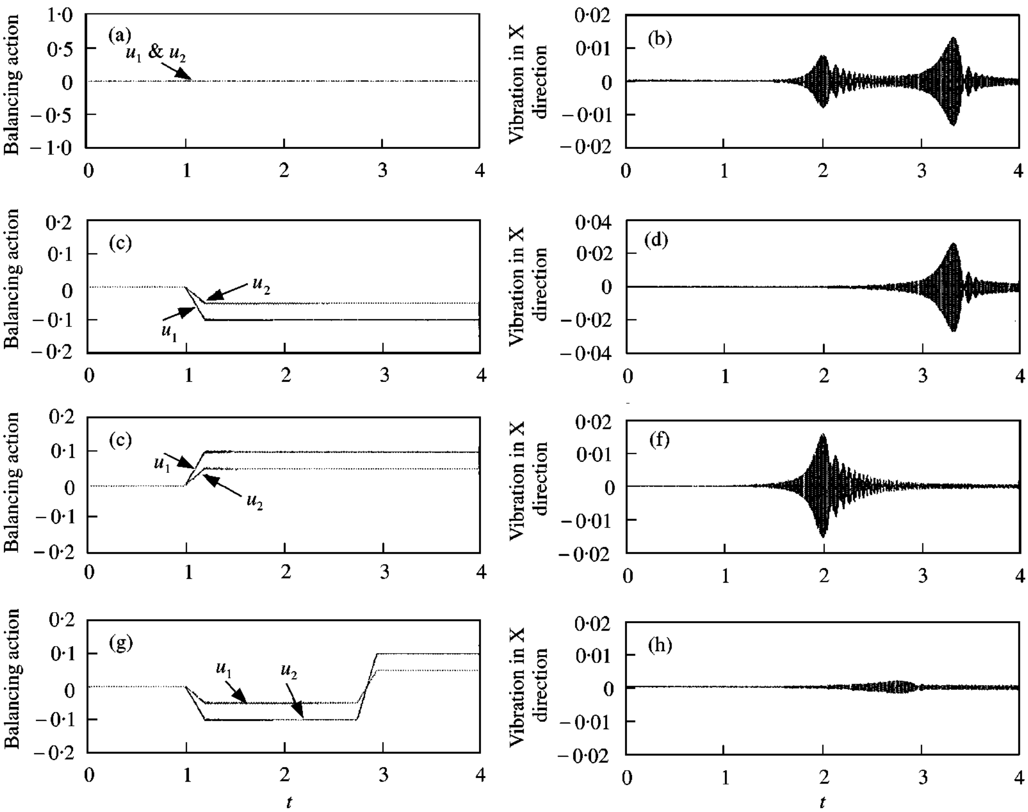

Figure 4. The results of different control strategies: (a), (c), (e), (g) different balancing strategies; (a) no balancing; (c) only balance the first vibration mode; (e) only balance the second vibration mode; (g) balance both vibration modes with the optimal switching function; (b), (d), (f), (h) the corresponding vibration output.

4. CONCLUSIONS

An adaptive active balancing strategy for a rotor system is proposed. By using this strategy, the imbalance-induced vibration during the acceleration period can be efficiently suppressed.

There are two basic challenges in this scheme. The first one is the estimation of the system imbalance during the acceleration period. This problem is solved by using an ordinary recursive least-squares estimation method based on a time domain model of the rotor system.

The second problem is also related to the acceleration. Since there are two or more vibration modes during acceleration in general, an optimal balancing strategy is required to suppress all the modes. This problem is solved based on an analytical expression of the imbalance response of a Jeffcott rotor during acceleration. Based on that expression, the optimal balancing strategy is found to be a simple step function. Confirmed by the simulation study, this strategy can reduce the imbalance-induced vibration efficiently.

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APPENDIX A: OUTLINE OF THE ANALYTICAL SOLUTION

The equation of motion of Jeffcott rotor is

$$\ddot{\mathbf{r}} + \mathbf{2}\varsigma\omega_n\dot{\mathbf{r}} + \omega_n^2\mathbf{r} = \mathbf{w}(\omega^2 - \mathrm{i}\alpha)\mathrm{e}^{\mathrm{i}\varphi}, \tag{A1}$$

where $\mathbf{r}$ is the position vector of the geometric center of the Jeffcott rotor, ζ and ω_n are the damping ratio and natural frequency of the rotor, $\mathbf{w}$ represents the system imbalance, ω is the rotating speed of the rotor, α is angular acceleration and φ is the rotating angle. If only the real part is considered,

$$\ddot{x} + 2\varsigma\omega_n\dot{x} + \omega_n^2 x = C_1\cos\left(\tfrac{1}{2}\,\alpha t^2 + \sigma'\right) + C_2 t^2\cos\left(\tfrac{1}{2}\alpha t^2 + \sigma\right), \tag{A2}$$

where C_1 and C_2 carry the magnitude information and σ and σ' carry the phase information of the imbalance. For a second order system, the response to an arbitrary forcing function is

$$x = \frac{1}{m(\alpha_0 - \beta_0)}\left\{\mathrm{e}^{\alpha_0 t}\int_0^t f(t)\mathrm{e}^{-\alpha_0 t}\,\mathrm{d}t - \mathrm{e}^{\beta_0 t}\int_0^t f(t)\mathrm{e}^{-\beta_0 t}\,\mathrm{d}t\right\}, \tag{A3}$$

where $\alpha_0 = -\zeta\omega_n + \mathrm{i}\sqrt{(\omega_n^2 - \zeta^2\omega_n^2)}$ $\beta_0 = -\zeta\omega_n - \mathrm{i}\sqrt{(\omega_n^2 - \zeta^2\omega_n^2)}$. To evaluate this integration expression, a variable substitution is applied.

$$z_1(t) = -\frac{\mathrm{i}}{2}(\alpha t - \mathrm{i}\omega_n\zeta - \omega_n\sqrt{1-\zeta^2})^2/\alpha, \qquad z_2(t) = -\frac{\mathrm{i}}{2}(\alpha t - \mathrm{i}\omega_n\zeta + \omega_n\sqrt{1-\zeta^2})^2/\alpha. \tag{A4}$$

Based on the Cauchy residue theorem, the whole vibration can be written in three parts.

A. 1. TRANSIENT RESPONSE DUE TO THE INITIAL CONDITION OF THE ROTOR

$$x_t = \mathrm{Re}\left[\frac{C_2 e^{i\sigma}}{m(\alpha_0-\beta_0)}\{e^{\alpha_0 t - z_1'}[C_4\psi(z_1') + C_5 + C_6\phi(z_1')] - e^{\beta_0 t - z_2'}[C_8\psi(z_2') + C_9 + C_{10}\phi(z_2')]\}\right]$$
$$+ \mathrm{Re}\left[\frac{C_1 e^{i\sigma}}{m(\alpha_0-\beta_0)\sqrt{2i\alpha}}\{e^{\alpha_0 t - z_1'} C_3\phi(z_1') - e^{\beta_0 t - z_2'} C_7\phi(z_2')\}\right]. \quad \text{(A5)}$$

A.2. SYNCHRONOUS VIBRATION

$$x_s = \mathrm{Re}\left[\frac{C_2 e^{i\sigma}}{m(\alpha_0-\beta_0)}\{-e^{\alpha_0 t - z_1}[C_4\psi(z_1) + C_5 + C_6\phi(z_1)] + e^{\beta_0 t - z_2}[C_8\psi(z_2) + C_9 + C_{10}\phi(z_2)]\}\right]$$
$$+ \mathrm{Re}\left[\frac{C_1 e^{i\sigma'}}{m(\alpha_0-\beta_0)\sqrt{2i\alpha}}\{-e^{\alpha_0 t - z_1} C_3\phi(z_1) + e^{\beta_0 t - z_2} C_7\phi(z_2)\}\right]. \quad \text{(A6)}$$

A.3. SUDDENLY OCCURRING TRANSIENT VIBRATION

$$x_n = \mathrm{Re}\left[\frac{C_2 e^{i\sigma}}{m(\alpha_0-\beta_0)} e^{\alpha_0 t}(\sqrt{\pi}KC_4 + 2\sqrt{\pi}KC_6) + \frac{C_1 e^{i\sigma'}}{m(\alpha_0-\beta_0)\sqrt{2i\alpha}}(2\sqrt{\pi}KC_3 e^{\alpha_0 t})\right]. \quad \text{(A7)}$$

Among the above equations, $\psi(z) = \int_0^\infty e^{-v}\sqrt{z+v}\,dv$, $\phi(z) = \int_0^\infty (e^{-v}/\sqrt{z+v})\,dv$,

$$K = \begin{cases} 0, & \text{when } t \leqslant \omega_n(\zeta + \sqrt{1-\zeta^2})/\alpha, \\ 1, & \text{when } t > \omega_n(\zeta + \sqrt{1-\zeta^2})/\alpha, \end{cases}$$

and C_1–C_{10} are complex constants.

APPENDIX B: NOMENCLATURE

m	mass of the shaft
k, c	stiffness and damping coefficient of bearings
m_u, m_b	imbalance mass of the rotor and imbalance mass provided by the balancer
u_x, u_y, u_z	co-ordinates (in body-fixed co-ordinate) of the imbalance
w_x, w_y, w_z	co-ordinates (in body-fixed co-ordinate) of the imbalance provided by the balancer
L, I_p, I_t	length, polar moment of inertia of mass and diametric moment of inertia of mass of the shaft
R_X, R_Z	displacements (in inertial co-ordinate) of the mass center of the shaft in X and Z directions
$\dot{R}_X$, $\dot{R}_Z$	velocities of the mass center of the shaft
$\ddot{R}_X$, $\ddot{R}_Z$	accelerations of the mass center
ϕ, $\dot{\phi}$, $\ddot{\phi}$	rotating angle, rotating speed, and rotating angular acceleration of the shaft
ψ, θ	Euler angles to describe the orientation of the body-fixed co-ordinate frame in the inertial co-ordinate frame. (ψ, θ, ϕ) forms an Euler angle set.