# The Analytical Imbalance Response of Jeffcott Rotor During Acceleration

**Shiyu Zhou**
Department of Mechanical Engineering,
The University of Michigan,
Ann Arbor, MI 48109

**Jianjun Shi***
Department of Industrial
and Operations Engineering,
The University of Michigan,
Ann Arbor, MI 48109
e-mail: shihang@umich.edu

*Since many rotor systems normally operate above their critical speeds, the problem of accelerating the machine through its critical speeds without excessive vibration draws increasing attention. This paper provides an analytical imbalance response of the Jeffcott rotor under constant acceleration. The response consists of three parts: transient vibration due to the initial condition of the rotor, ''synchronous'' vibration, and suddenly occurring vibration at the damped natural frequency. This solution provides physical insight to the vibration of the rotor during acceleration.* [DOI: 10.1115/1.1352021]

## 1 Introduction

The characteristics of the transition vibration of a rotor system when it passes its critical speeds during acceleration are of great interest for active vibration control, active real-time balancing [1], and rotor design. In the past, a few analyses [2–8] have dealt with speed varying transient rotor dynamics. These researchers used numerical integration techniques to calculate numerical solutions to the transient dynamic model. Although these models can be used to predict the transient vibration for a complicated rotor system, it remains hard to obtain the quantitative characteristics of the transient vibration.

Lewis [9] and Dimentberg [10] presented an analytical solution of the problem of running a rotor system through its critical speeds at a uniform acceleration. The basic characteristic of the ''envelope'' (amplitude) of the transient vibration was studied by an approximation method. In this paper, their work is extended. An analytical expression of the motion of the geometric center of a simple Jeffcott rotor is derived. The exact ''envelope'' and ''phase'' of the transient vibration are presented. As stated in Dimentberg [10], it is found that the transient vibration through critical speeds consists of free vibration and synchronous vibration. Explicit expressions of these two components are presented in this paper.

## 2 Problem Statement

The Jeffcott rotor is a simplified rotor model that retains the essential characteristics of more realistic rotor models in its imbalance response. The geometric setup of this model is shown in Fig. 1. In this setup, the bearings are rigid and frictionless. The shaft is isotropic, elastic, and massless. The disk is rigid and is located at the center of the shaft.

In practice, only the motion of the geometric center of the disk can be easily measured. Therefore, we take point $S$ as the interested point. The equation of motion of point $S$ in complex form is [11]

$$\ddot{\mathbf{r}}+2\varsigma\omega_n\dot{\mathbf{r}}+\omega_n{}^2\mathbf{r}=\mathbf{w}(\omega^2-i\alpha)e^{i\varphi}. \tag{1}$$

Since the stationary coordinate O$xy$ and the body-fixed coordinate systems (they coincide with each other at the initial condition) are usually selected arbitrarily, the system imbalance is represented as a general vector **w**, instead of just an eccentricity. The real part and the imaginary part are symmetric in the above equation because the shaft is isotropic and there is no gyroscopic effect in this rotor model. Therefore, the vibration characteristic of this system can be obtained by only considering this second order model in one direction. It is well known that the higher order motion of a general flexible rotor can be viewed as a summation of several second order motions by modal analysis technique if the gyroscopic effect is not significant. Hence, the result in this paper can also be applied to complicated flexible rotor models.

To simplify the problem, we assume that the acceleration is constant. The real part of the governing equation changes to

$$\ddot{x}+2\varsigma\omega_n\dot{x}+\omega_n{}^2x=C_1\cos(\alpha t^2/2+\sigma')+C_2t^2\cos(\alpha t^2/2+\sigma). \tag{2}$$

For a second order system, the response to an arbitrary forcing function $f(t)$ is [12]

$$x=\left\{e^{\alpha_0 t}\int_0^t f(\tau)e^{-\alpha_0\tau}d\tau-e^{\beta_0 t}\int_0^t f(\tau)e^{-\beta_0\tau}d\tau\right\}/(m(\alpha_0-\beta_0)). \tag{3}$$

In the next section, an equivalent analytical expression for this integration solution will be presented.

## 3 Derivation of the Analytical Solution

The system is a linear system. If we let $x_1$ be the response corresponding to the first forcing term and $x_2$ be the response corresponding to the second forcing term, the total solution is the summation of these two.

- **The response caused by** $C_2t^2\cos(\alpha t^2/2+\sigma)$

  Noting $\cos(\alpha t^2/2+\sigma)=\mathrm{Re}[e^{i(\alpha t^2/2+\sigma)}]$, the response is

$$x_2=\mathrm{Re}\left[C_2e^{i\sigma}\left\{e^{\alpha_0 t}\int_0^t\tau^2e^{i\alpha\tau^2/2}e^{-\alpha_0\tau}d\tau-e^{\beta_0 t}\times\int_0^t\tau^2e^{i\alpha\tau^2/2}e^{-\beta_0\tau}d\tau\right\}/(m(\alpha_0-\beta_0))\right]. \tag{4}$$

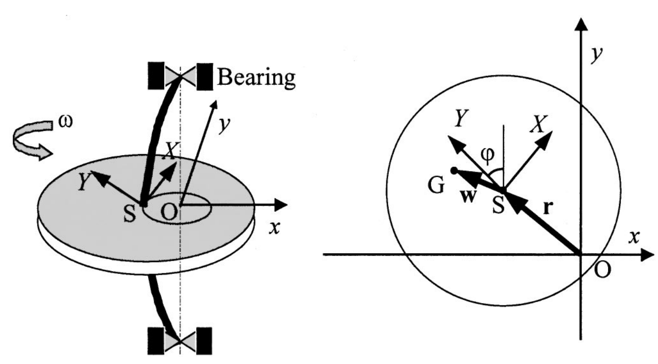


**Fig. 1 The geometric setup of a planar rotor**

*Author to whom all correspondence should be addressed.

Contributed by the Manufacturing Engineering Division for publication in the JOURNAL OF MANUFACTURING SCIENCE AND ENGINEERING. Manuscript received April 1999; revised Feb. 2000. Associate Editor. R. Furness.

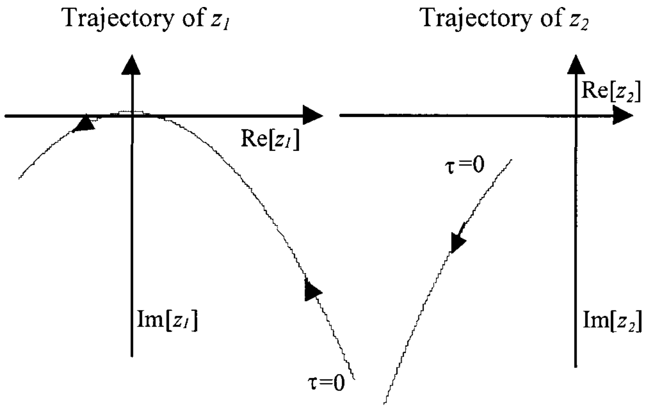


**Fig. 2 The trajectories of $z_1$ and $z_2$**

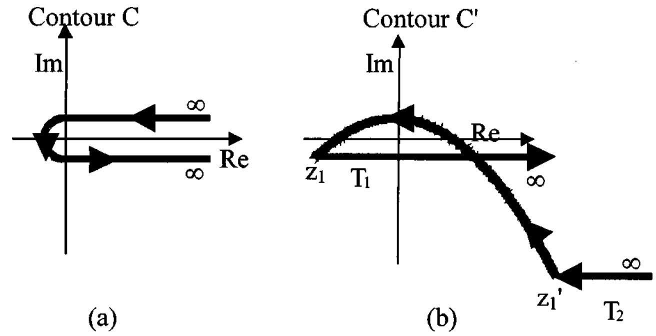


**Fig. 3 The integration curves**

Using the variable substitution [9], $\tau=\sqrt{2iz_1/\alpha}+(i\omega_n\zeta+\omega_n\sqrt{1-\zeta^2})/\alpha$ for term $\int_0^t\tau^2e^{i\alpha\tau^2/2}e^{-\alpha_0\tau}d\tau$ and $\tau=\sqrt{2iz_2/\alpha}+(i\omega_n\zeta-\omega_n\sqrt{1-\zeta^2})/\alpha$ for term $\int_0^t\tau^2e^{i\alpha\tau^2/2}e^{-\beta_0\tau}d\tau$, the integration along the real axis in Eq. (4) transforms to an integration along a curve in the complex plane. The two curves are defined by the trajectory of $z_1$ and $z_2$. $z_1$ and $z_2$ are

$$z_1(\tau)=-i(\alpha\tau-i\omega_n\zeta-\omega_n\sqrt{1-\zeta^2})^2/(2\alpha)$$
$$z_2(\tau)=-i(\alpha\tau-i\omega_n\zeta+\omega_n\sqrt{1-\zeta^2})^2/(2\alpha). \tag{5}$$

Typical trajectories for $z_1$ and $z_2$ with $\omega_n=100\,\mathrm{s}^{-1}$, $\zeta=0.01$, and $\alpha=15\,\mathrm{s}^{-2}$ are shown in Fig. 2.

Both the trajectories are in the form of a parabola. Using this variable substitution, the first term in the bracket of Eq. (4) turns out to be

$$\int_0^t\tau^2e^{i\alpha\tau^2/2}e^{-\alpha_0\tau}d\tau=ie^{(-i\omega_n^2+2i\omega_n^2\zeta^2+2\omega_n^2\zeta\sqrt{1-\zeta^2})/(2\alpha)}\times\int_0^t(\sqrt{2iz_1/\alpha}-\alpha_0 i/\alpha)^2e^{-z_1}/\sqrt{2i\alpha z_1}\,dz_1. \tag{6}$$

Equation (6) can be simplified by using constants $C_3\sim C_6$.

$$\int_0^t\tau^2e^{i\alpha\tau^2/2}e^{-\alpha_0\tau}d\tau=C_4\int_0^te^{-z_1}\sqrt{z_1}dz_1+C_5\int_0^te^{-z_1}dz_1+C_6\int_0^te^{-z_1}/\sqrt{z_1}dz_1. \tag{7}$$

From Macrobert [13],

$$\oint_C e^{-z}z^{x-1}dz=(e^{2\pi xi}-1)\Gamma(x), \tag{8}$$

where $\Gamma(x)$ is the gamma function and the integration curve $C$ is shown in Fig. 3($a$).

Noting that the origin (0,0) is the only singular point of the complex variable function $e^{-z}z^{x-1}$ when $x=1/2$ or $x=3/2$, by the Cauchy residue theorem,

$$\oint_{C'}e^{-z}z^{x-1}dz=\begin{cases}0 & \text{when } C' \text{ does not include the origin}\\ -\oint_C e^{-z}z^{x-1}dz & \text{when } C' \text{ includes the origin.}\end{cases} \tag{9}$$

The contour $C'$ is shown in Fig. 3($b$). Noting $\Gamma(1/2)=\sqrt{\pi}$, $\Gamma(3/2)=\sqrt{\pi}/2$, Eqs. (8), and (9), yields

$$\oint_{C'}e^{-z}\sqrt{z}dz=\begin{cases}0 & \text{when } C' \text{ does not include the origin}\\ \sqrt{\pi} & \text{when } C' \text{ includes the origin}\end{cases}$$
$$\oint_{C'}\frac{e^{-z}}{\sqrt{z}}dz=\begin{cases}0 & \text{when } C' \text{ does not include the origin}\\ 2\sqrt{\pi} & \text{when } C' \text{ includes the origin.}\end{cases} \tag{10}$$

Let $z_1'=z_1(t=0)=-i\omega_n^2(i\zeta-\sqrt{1-\zeta^2})^2/(2\alpha)$, and the integration from $z_1'$ to $z_1$ is

$$\int_{Curve:\text{from } z_1' \text{ to } z_1}e^{-z}z^{x-1}dz=\oint_{C'}e^{-z}z^{x-1}dz-\int_{T_1}e^{-z}z^{x-1}dz-\int_{T_2}e^{-z}z^{x-1}dz, \tag{11}$$

where the trajectory $T_1$ and $T_2$ are shown in Fig. 3($b$). The right-hand side of Eq. (7) changes to

$$C_4[\sqrt{\pi}K+e^{-z_1'}\psi(z_1')-e^{-z_1}\psi(z_1)]+C_5[e^{-z_1'}-e^{-z_1}]+C_6[2\sqrt{\pi}K+e^{-z_1'}\phi(z_1')-e^{-z_1}\phi(z_1)] \tag{12}$$

where $\psi(z)=\int_0^\infty e^{-v}\sqrt{z+v}dv$, $\phi(z)=\int_0^\infty e^{-v}/\sqrt{z+v}dv$, which are obtained by a simple variable substitution, and $K$ is an indicator function, which indicates if the contour of $C'$ includes the origin, i.e., if $z_1$ is in the 3rd quadrant. More clearly,

$$K=\begin{cases}0 & \text{when } t\leqslant\omega_n(\zeta+\sqrt{1-\zeta^2})/\alpha\\ 1 & \text{when } t>\omega_n(\zeta+\sqrt{1-\zeta^2})/\alpha.\end{cases} \tag{13}$$

The second term in the bracket of Eq. (4) is simpler than the first term because all the trajectory of $z_2$ is in the 3rd quadrant. No singular point is included in the integration contour. With a similar derivation, we can get

$$\int_0^t\tau^2e^{i\alpha\tau^2/2}e^{-\beta_0\tau}d\tau=C_8[e^{-z_2'}\psi(z_2')-e^{-z_2}\psi(z_2)]+C_9(e^{-z_2'}-e^{-z_2})+C_{10}[e^{-z_2'}\phi(z_2')-e^{-z_2}\phi(z_2)]. \tag{14}$$

Substituting Eqs. (12) and (14) into Eq. (4), we obtain an analytical expression for $x_2$.

$$x_2 = \mathrm{Re}[C_2\{e^{\alpha_0 t + i\sigma}[C_4(\sqrt{\pi}K + e^{-z_1'}\psi(z_1') - e^{-z_1}\psi(z_1)) + C_5(e^{-z_1'} - e^{-z_1}) + C_6(2\sqrt{\pi}K + e^{-z_1'}\phi(z_1') - e^{-z_1}\phi(z_1))] - e^{\beta_0 t + i\sigma}[C_8(e^{-z_2'}\psi(z_2') - e^{-z_2}\psi(z_2)) + C_9(e^{-z_2'} - e^{-z_2}) + C_{10}(e^{-z_2'}\phi(z_2') - e^{-z_2}\phi(z_2))]\}/(m(\alpha_0 - \beta_0))] \quad (15)$$

• **The response caused by** $C_1 \cos(\alpha t^2/2 + \sigma')$

The integration form for the response corresponding to $C_1 \cos(\alpha t^2/2 + \sigma')$ is

$$x_1 = \mathrm{Re}\left[C_1 e^{i\sigma'}\left\{e^{\alpha_0 t}\int_0^t e^{i\alpha\tau^2/2}e^{-\alpha_0\tau}d\tau - e^{\beta_0 t}\int_0^t e^{i\alpha\tau^2/2}e^{-\beta_0\tau}d\tau\right\}/(m(\alpha_0 - \beta_0))\right]. \quad (16)$$

Using the same variable substitution as Eq. (5) and by similar derivation, we can get the response,

$$x_1 = \mathrm{Re}[C_1 e^{i\sigma'}\{e^{\alpha_0 t}C_3(2\sqrt{\pi}K + e^{-z_1'}\phi(z_1') - e^{-z_1}\phi(z_1)) - e^{\beta_0 t}C_7(e^{-z_2'}\phi(z_2') - e^{-z_2}\phi(z_2))\}/(m(\alpha_0 - \beta_0)\sqrt{2i\alpha})]. \quad (17)$$

The total imbalance response of the Jeffcott model during acceleration can be obtained by the summation of $x_1$ and $x_2$ as given in Eqs. (17) and (15).

## 4 Analysis

Although numerical integration has to be used to calculate functions $\phi(z)$ and $\psi(z)$ in Eqs. (17) and (15), the analytical solution can provide more insights to the imbalance response during acceleration than a direct numerical solution. From the analytical solution, the full response can be viewed as three parts: a transient response due to initial condition; a ''synchronous'' vibration; and a suddenly occurring vibration at the damped natural frequency.

• **The transient response is contributed by the terms** $\psi(z_i')$, $\phi(z_i')$, **and** $e^{z_i'}$, $i=1,2$.

$$x_t = \mathrm{Re}[[C_2 e^{i\sigma}\{e^{\alpha_0 t - z_1'}[C_4\psi(z_1') + C_5 + C_6\phi(z_1')] - e^{\beta_0 t - z_2'}[C_8\psi(z_2') + C_9 + C_{10}\phi(z_2')]\} + C_1 e^{i\sigma'}\{e^{\alpha_0 t - z_1'}C_3\phi(z_1') - e^{\beta_0 t - z_2'} \times C_7\phi(z_2')\}/\sqrt{2i\alpha}]/(m(\alpha_0 - \beta_0))]. \quad (18)$$

In the transient response, only $e^{\alpha_0 t}$ and $e^{\beta_0 t}$ are time variant. The response vibrates at the damped natural frequency $\omega_n\sqrt{1-\zeta^2}$ and decays at the rate of $e^{-\zeta\omega_n}$.

• **The ''synchronous'' vibration is contributed by the terms** $\psi(z_i)$, $\phi(z_i)$, $i=1,2$.

$$x_s = \mathrm{Re}[[C_2 e^{i\sigma}\{-e^{\alpha_0 t - z_1}[C_4\psi(z_1) + C_5 + C_6\phi(z_1)] + e^{\beta_0 t - z_2}[C_8\psi(z_2) + C_9 + C_{10}\phi(z_2)]\} + C_1 e^{i\sigma'}\{-e^{\alpha_0 t - z_1}C_3\phi(z_1) + e^{\beta_0 t - z_2}C_7\phi(z_2)\}/\sqrt{2i\alpha}]/(m(\alpha_0 - \beta_0))]. \quad (19)$$

From Eq. (5), we have

$$e^{\alpha_0 t - z_1(t)} = e^{-\omega_n^2\zeta\sqrt{1-\zeta^2}/\alpha + [\alpha t^2/2 + \omega_n^2(1-2\zeta^2)/(2\alpha)]i}$$
$$e^{\beta_0 t - z_2(t)} = e^{\omega_n^2\zeta\sqrt{1-\zeta^2}/\alpha + [\alpha t^2/2 + \omega_n^2(1-2\zeta^2)/(2\alpha)]i}. \quad (20)$$

This component is called ''synchronous'' vibration because the instantaneous vibration frequency of $x_s$ is $\alpha t$, which is obtained by the time derivative of $\alpha t^2/2 + \omega_n^2(1-2\zeta^2)/(2\alpha)$. It is the same frequency as the instantaneous rotating speed.

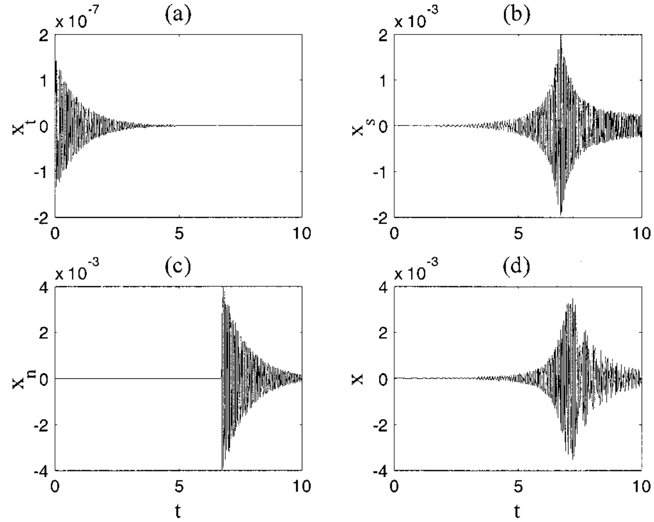


**Fig. 4 Transient imbalance response calculated by the analytical solution**

• **The suddenly occurring vibration with the damped natural frequency is contributed by** $2\sqrt{\pi}K$ **and** $\sqrt{\pi}K$.

The equation for this component is

$$x_n = \mathrm{Re}[\{C_2 e^{i\sigma + \alpha_0 t}(\sqrt{\pi}KC_4 + 2\sqrt{\pi}KC_6) + 2\sqrt{\pi}KC_1C_3 e^{\alpha_0 t + i\sigma'}/\sqrt{2i\alpha}\}/(m(\alpha_0 - \beta_0))]. \quad (21)$$

It occurs when $K$ is first nonzero, i.e., when $t = \omega_n(\zeta + \sqrt{1-\zeta^2})/\alpha$. The occurring time is very close to the time when the rotating speed hits the damped natural frequency $\omega_n\sqrt{1-\zeta^2}/\alpha$ when the damping of the system is low. It is also exponential decaying because the only vibrating term is $e^{\alpha_0 t}$ and $e^{\beta_0 t}$.

Figure 4 shows these vibration components and the full response for $\omega_n = 100\ \mathrm{s}^{-1}$, $\zeta = 0.01$, $\alpha = 15\ \mathrm{s}^{-2}$, and $\mathbf{w} = (1+i) \times 10^{-5}$.

The transient response (Fig. 4(*a*)) is quite small compared to the full response. If the transient vibration is ignored, there are two major components in the response: the synchronous vibration (Fig. 4(*b*)) and the exponential decaying vibration at the damped natural frequency after time $t = \omega_n(\zeta + \sqrt{1-\zeta^2})/\alpha$ (Fig. 4(*c*)). Although the function $K$ is discontinuous, the full response is continuous because the function $\phi(z)$ and $\psi(z)$ are also discontinuous. These discontinuous functions compensate each other. The full analytical response (Fig. 4(*d*)) has been compared with the numerical solution of Eq. (1) that is calculated by the Runge-Kutta method. The relative difference is less than 0.5 percent at each time step.

Another useful observation of this analytical solution is that if only $x_s$ and $x_n$ are considered, the transient response for a general rotor system without gyroscopic effects can be written as

$$\mathbf{v} = \sum_{k=1}^{N}[\mathbf{w}_k e^{i[\alpha t^2/2 + \rho_k(t)]}M_{sk}(t) + \mathbf{w}_k e^{i[\omega_{dk}t + \gamma_k(t)]}M_{nk}(t)]. \quad (22)$$

$\mathbf{v}$ is a complex number that represents the vibration of the rotor in two directions. N is the number of significant vibration modes. $\mathbf{w_k}$ is a complex number that represents the system imbalance in the $k$th mode. $\omega_{dk}$ is the damped natural frequency of the $k$th mode. $M_{sk}$, $\rho_k$, $M_{nk}$, and $\gamma_k$ are defined as the generic magnitudes and phases of the synchronous vibration and the suddenly occurring vibration in the $k$th mode, respectively. They are only related with the dynamic parameters of the rotor system, not with the system

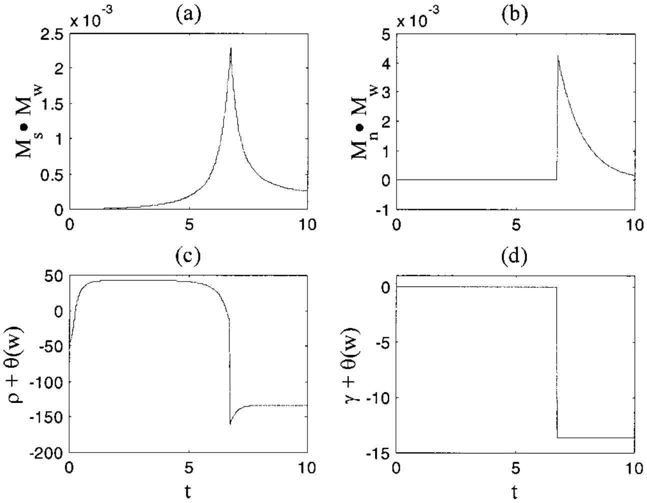


**Fig. 5 The magnitudes and phases of $x_s$ and $x_n$**

imbalance. For a simple Jeffcott rotor, there is only one vibration mode. Using the same parameters as that used in Fig. 4, we give an example of the magnitudes and phases of the vibration components in Fig. 5. $M_s$, $\rho$, and $M_n$, $\gamma$ are the generic magnitudes and phases of synchronous and suddenly occurring vibration, respectively. $M_{\mathbf{w}}$ and $\theta(\mathbf{w})$ are the magnitude and phase of the system imbalance. The unit for phase in Fig. 5 is degree. This information is very useful for design of active vibration control and estimation of magnitude and position of imbalance.

## 5 Conclusion

In this paper, an analytical solution for the imbalance response of the Jeffcott rotor during acceleration is obtained. This solution provides physical insights into the motion of a Jeffcott rotor during constant acceleration. The solution quantitatively shows that the motion consists of three parts: a transient vibration at damped natural frequency, a synchronous vibration with the frequency of instantaneous ''synchronous'' frequency, and a suddenly occurring vibration at damped natural frequency. This analytical solution provides guidelines for rotor design, estimation of system imbalance, synthesis of active vibration control, and balancing of rotor systems.

## Nomenclature

$C_1 \sim C_{10}$ = constants used to simplify the expression

$C_1 = -\alpha\sqrt{U_i^2+U_r^2} \quad C_2=\alpha^2\sqrt{U_i^2+U_r^2}$

$C_3 = ie^{(-i\omega_n^2+2i\omega_n^2\zeta^2+2\omega_n^2\zeta\sqrt{1-\zeta^2})/(2\alpha)}$

$C_4 = C_3\sqrt{2i}/(\alpha\sqrt{\alpha}) \quad C_5=-2\alpha_0 iC_3/\alpha^2$

$C_6 = -\alpha_0^2 C_3/(\alpha^2\sqrt{2i\alpha})$

$C_7 = ie^{-(i\omega_n^2-2\omega_n^2\zeta^2+2\omega_n^2\zeta\sqrt{1-\zeta^2})/(2\alpha)} \quad C_8=C_7\sqrt{2i}/(\alpha\sqrt{\alpha})$

$C_9 = -2C_7\beta_0 i/\alpha^2 \quad C_{10}=-C_7\beta_0^2/(\alpha^2\sqrt{2i\alpha})$

$M_w, \theta(w)$ = magnitude and phase of system imbalance in Fig. 5.

$Oxy$ = stationary coordinate system

$S$, $G$ = geometric and mass centers of the disk

$SXY$ = body-fixed coordinate system

$U_r, U_i$ = real and imaginary part (the $X$ and $Y$ coordinates) of $\mathbf{w}$

$i = \sqrt{-1}$

$\mathbf{r}$ = $x+iy$, the vector from $O$ to $S$ in stationary coordinate system

$\mathbf{w}$ = $U_r+iU_i$, a constant vector from S to G in body-fixed coordinate system

$z_1, z_2$ = functions of time $t$ or dummy variable $\tau$ when they are in the integrand

$z_1', z_2'$ = complex values of $z_1$ and $z_2$ at $t=0$

$\varphi$, $\omega$, $\alpha$ = rotating angle, speed and acceleration of the rotor

$\alpha_0 = -\zeta\omega_n + i\sqrt{(\omega_n^2-\zeta^2\omega_n^2)}$

$\beta_0 = -\zeta\omega_n - i\sqrt{(\omega_n^2-\zeta^2\omega_n^2)}$

$\zeta, \omega_n$ = damping ratio and natural frequency of the rotor system

$\sigma$ and $\sigma'$ = $\arctan U_i/U_r$ and $\arctan U_i/U_r+\pi/2$

$\tau$ = dummy variable for integration


## Acknowledgment

This work was supported by BalaDyne Corporation and the U.S. Department of Commerce, National Institute of Standards and Technology, Advanced Technology Program, Cooperative Agreement Number 70NANB7H3029.


## References


[1] Dyer, S. W., 1999, ''Adaptive Optimal Control of Active Balancing Systems for High-speed Rotating Machinery,'' Ph.D. Dissertation, University of Michigan.

[2] Shen, F. A., 1972, ''Transient Flexible-Rotor Dynamics Analysis, Part 1-Theory,'' ASME J. Eng. Ind., May, pp. 531–538.

[3] Childs, D., 1969, ''Simulation Models for Flexible Spinning Bodies,'' Simulation, June, pp. 291–296.

[4] Childs, D., 1972, ''A Simulation Model for Flexible Rotating Equipment,'' ASME J. Eng. Ind., Feb., pp. 201–209.

[5] Childs, D., 1976, ''A Modal Transient Simulation Model for Flexible Asymmetric Rotors,'' ASME J. Eng. Ind., Feb., pp. 312–318.

[6] Subbiah, R., and Rieger, N. F., 1988, ''On the Transient Analysis of Rotor-Bearing Systems,'' ASME J. Vibr. Acoust., **110**, Oct. pp. 515–520.

[7] Subbiah, R., Kumar, A. S., and Sankar, T. S., 1988, ''Transient Dynamic Analysis of Rotors Using the Combined Methodologies of Finite Elements and Transfer Matrix,'' ASME J. Appl. Mech., **55**, June, pp. 448–452.

[8] Lee, A. C., Kang, Y., Tsai, K. L., and Hsiao, K. M., 1992, ''Transient Analysis of an Asymmetric Rotor-Bearing System During Acceleration,'' ASME J. Eng. Ind., **114**, Nov., pp. 465–475.

[9] Lewis, F. M., 1932, ''Vibration During Acceleration Through a Critical Speed,'' ASME J. Appl. Mech., APM-54-24, pp. 253–261.

[10] Dimentberg, F. M., 1961, *Flexural Vibrations Of Rotating Shafts*, Butterworth, London.

[11] Genta, G., 1993, *Vibration of Structures and Machines: Practical Aspects*, Springer-Verlag, New York.

[12] Newland, D. E., 1989, *Mechanical Vibration Analysis and Computation*, Longman Scientific & Technical, Burnt Mill.

[13] Macrobert, T. M., 1954, *Functions of a Complex Variable*, Macmillan, London.